Deformations of Non - Isolated Singularities.

by

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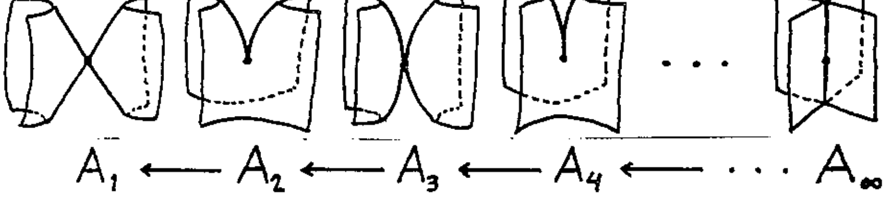
Contents :

Introduction.

Given a germ X = (X,p) of an analytic space with an isolated singular point p, one has a *semi-universal deformation* $\mathfrak{X} \longrightarrow B$. It has the property that all flat families over a space Z with X as special fibre are induced by a map $Z \longrightarrow B$, which is unique on the level of tangent spaces. The space B and the deformation $\mathfrak{X} \longrightarrow B$ are unique, but only up to non-canonical isomorphism (see [Gra], [Pa], [Schl1], [Bi]). The space B is called the *base space (of the semi-universal deformation) of X*. If X is a hypersurface, or more generally a complete intersection, then B is smooth (see [Tj1]). If X is Cohen-Macaulay of codimension 2 (i.e. embdim(X) - dim(X) = 2), then B is also smooth (see [Scha]). In general however B may be singular and even have components of different dimensions. The simplest example is the cone over the rational normal curve of degree 4 in $\mathbb{P}^4$, due to Pinkham (see [Pi]). In this case B consists of two smooth components, one of dimension three and one of dimension one, intersecting each other transversally.

In general it is very hard to compute the base space B for a given singularity X. Only recently the base spaces of all cyclic quotient singularities were determined by Arndt (see [Arn]). Usually the first step in the construction of B consists of finding T^1_X , the set of first order deformations of X, which can be naturally identified with the Zariski tangent space of B. This space T^1_X has received much attention, in particular in the case that X is a normal surface singularity. Using a resolution of X one can try to compute T^1_X in terms of of resolution data (see [La], [Wa1]). This has been rather succesful for rational singularities (see [Ri], [B-K]).

For a hypersurface singularity T^1_X can be identified with $\mathcal{O}_X/J(f)$, where J(f) is the the ideal generated by the partial derivatives of f, and f = 0 is a defining equation of X. One easily sees that T^1_X is finite dimensional if and only if X has an isolated singular point. When X does not have an isolated singular point, it is natural to look for a *special class* of deformations, namely the class of deformations for which the singular locus of X is deformed flatly (and stays inside the singular locus of the deformed X!). Under appropriate circumstances one can hope for a finite dimensional base space, because the infinite dimensionality of T^1_X is caused by the 'opening up' of the singularities transverse to the singular locus. A good example to keep in mind is the A_k-series of deformations of the A_∞-singularity:



In his thesis, Pellikaan [Pe1] (see also [Pe2]) started with a theory along these lines, extending the case that the singular locus is smooth and one dimensional, which was considered by Siersma (see [Si]). Pellikaan's main results however concern the case that the singular locus Σ is a complete intersection or the case that Σ is not deformed at all. If Σ is a complete intersection, then the base space of the functor considered by Pellikaan is smooth (see also §3.C). That this is not always the case, can be seen by the following beautiful example of Pellikaan (see [Pe 2], ex.2.4). This is the example of $f = (yz)^2 + (zx)^2 + (xy)^2$. Here the (reduced) singular locus Σ is described by the ideal $I = (\Delta_1, \Delta_2, \Delta_3) = (yz, zx, xy)$, so Σ consists of the coordinate axes in $\mathbb{C}^3$. He gives two types of deformations. First of all, one can deform the curve Σ, giving a deformed ideal $\tilde{I} = (\tilde{\Delta}_1, \tilde{\Delta}_2, \tilde{\Delta}_3)$. Then obviously $F = (\tilde{\Delta}_1)^2 + (\tilde{\Delta}_2)^2 + (\tilde{\Delta}_3)^2$ gives a deformation in the above sense. Because one knows that $\dim T^1_\Sigma = 3$, and in this case Σ is Cohen-Macaulay of codimension 2, one gets a deformation over a smooth 3 dimensional space. Another deformation, over a smooth curve with parameter s, is given by $F = f + s.xyz$. Here Σ is not deformed at all, and the deformation is also admissible in the above sense, because $xyz = 0$ has also the coordinate axes as singular locus. These two types of deformations are 'essentially' all admissible deformations of f. So we get a family over a space B which is the same as the base space in Pinkham's example we mentioned above. This is not a coincidence. Because our space X, defined by $f = 0$, has singularities in codimension one, it is not normal. Now the normalization $\tilde{X}$ of X is precisely the cone over the rational normal curve in $\mathbb{P}^4$. Moreover, the total space of the deformation over the one dimensional component of $\tilde{X}$ can be identified with the cone over the Veronese surface in $\mathbb{P}^5$ (see [Pi]). It is known that a 'generic' projection in $\mathbb{P}^3$ of the Veronese surface is the Steiner Roman surface, described in homogeneous coordinates by the equation $(yz)^2 + (zx)^2 + (xy)^2 + sxyz = 0$ (see [S-R], pp.128-135). This indeed corresponds exactly to the second type of deformation of X described above.

In general, we consider a normal surface singularity $\tilde{X}$, embedded in some high dimensional space. When we now project $\tilde{X}$ down to $\mathbb{C}^3$, we get a hypersurface X as image. This hypersurface X will in general have a curve Σ as singular locus. For a 'generic' projection X will be *weakly normal* or what is the same, X will have *transverse* A_1 *-singularities* meaning that in a general point $q \in \Sigma$ one has $(X,q) \approx A_\infty$ (i.e. Σ will be an ordinary double curve). Conversely, given a weakly normal surface $X \subset \mathbb{C}^3$, one can take the normalization to get an $\tilde{X}$. Now the statement is that the functor of admissible deformations of X is equivalent to the deformation functor of the diagram $\tilde{X} \longrightarrow X$. (see § 4.) As the deformation theory of $\tilde{X}$ is not 'essentially' different from the deformation theory of the diagram $\tilde{X} \longrightarrow X$, this implies that all pathologies occurring in the deformation theory of normal surfaces are reflected in the deformation theory of non-isolated hypersurface singularities in $\mathbb{C}^3$.

The purpose of this paper is to develop the (formal) theory of *admissible deformations* of non-isolated singularities, as intended above. We give a short overview of what to expect. In § 0. we treat some algebraic results. This paragraph should be used as a reference, and can therefore be skipped on first reading. In § 1. we introduce the functor of admissible deformations $Def(\Sigma,X)$ of a singularity X with a subspace Σ of the singular locus of X as a sub-functor of the deformations of the diagram $\Sigma \hookrightarrow X$, and investigate the Schlessinger conditions. In § 2. we consider the problem whether the natural forgetful transformation $Def(\Sigma,X) \longrightarrow Def(X)$ is injective. In § 3. we develop the infinitesimal deformation theory for non-isolated hypersurfaces. We determine the tangent space $T^1(\Sigma,X)$ of $Def(\Sigma,X)$ and identify the obstruction space $T^2(\Sigma,X)$. In § 4. we prove the above mentioned equivalence between $Def(\Sigma,X)$ and $Def(\tilde{X} \longrightarrow X)$. In § 5. finally we give examples and appplications. For a weakly normal surface X in $\mathbb{C}^3$ with normalization $\tilde{X}$ we give a formula for $T^1_{\tilde{X}}$ in terms of X only. Furthermore we prove a theorem about the dimension of the smoothing components of a normal surface singularity $\tilde{X}$ in terms of the number of triple points ($xyz = 0$) occurring in the deformation of X. Finally, in § 6. we determine, up to a smooth factor, the base spaces of the semi-universal deformation of all rational quadruple points. This proof reflects our experience that to understand the deformation theory of normal surface singularities it is essential to study the double locus Σ of a projection in $\mathbb{C}^3$ (c.f. (3.28)).

It should be stressed that although we try to formulate our results as general as possible, the case that interests us most and which we always *have in mind* is the case where X is an an analytic germ of a weakly normal surface in $\mathbb{C}^3$ and Σ is the singular locus of X, with its *reduced* structure. So along our way we are always willing to make any assumption on X and Σ as long as it applies to this case. For some results simpler arguments can be given in the special case we have in mind, which we usually for reasons of organization and clearity have avoided.

Conventions.

By a *space* we always mean an analytic space germ or the spectrum of a local ring. Typical names for spaces are X, Y, T, Σ, etc, for rings R, P, S, etc. When we say that 'X_S is a space over S' we mean that X_S is a space with a map to Spec(S) or to S, depending on whether S is a *ring* (this is usually the case) or a *space* . In such a relative situation we do simply write X_S/S in cases where one usually should write X_S/Spec(S). Although we are not completely systematic in this respect, we do not expect any confusion to arise.

Aknowledgement.

We would like to thank G.-M. Greuel, J. de Jong, D. Mond, R. Pellikaan, F.-O. Schreyer, J. Steenbrink, J. Stevens and all other people who helped us in one way or another. The first author was supported by the S.M.C with financial aid of the N.W.O. and the second by the Departement of Social Security of the Netherlands.

§ 0. *Algebraic Preliminaries*

In this preliminary paragraph we formulate some results from local commutative algebra which will be of use in § 2 and § 4 of this paper. Lacking a comprehensive reference and for convenience of the reader we include proofs. The results are centered around the interaction between the notion of flatness, base change properties of Ext, Cohen - Macaulay properties and duality. This paragraph may be skipped on a first reading.

In the sequel we adopt the following conventions:

* S is a noetherian local ring with maximal ideal m_S and residue field $k = S/m_S$.
* R is a noetherian local S - algebra
* We also consider R - modules M, N, . . . , and we always assume them to be finitely generated.
* When we put a bar over a module M we always mean $\overline{M} = \overline{R} \otimes_R M = k \otimes_S M$, where $\overline{R} = k \otimes_S R$.
* For some results we assume that R is a quotient of flat S - algebra P such that $\overline{P}$ is a regular local ring.

Proposition (0.1) :

Let M and N be two S-flat R - modules. Consider the natural mappings

$$\varphi_i \ : \ \mathrm{Ext}_R^i(M,N) \otimes k \longrightarrow \mathrm{Ext}_{\overline{R}}^i(\overline{M},\overline{N})$$

Then: i) If φ_i is surjective, then it is an isomorphism.

ii) If φ_i and φ_{i-1} are surjective, then $\mathrm{Ext}_R^i(M,N)$ is S-flat.

proof: This is a slight variation on the 'Cohomology and base change theorem' (see [Ha], pp 282-290). Let $F_\cdot \longrightarrow M$ be an R - free resolution of M. Then the complex $N^\cdot := \mathrm{Hom}_R(F_\cdot, N)$ consists of finitely generated R - modules which are S - flat and has $\mathrm{Ext}^i_R(M,N)$ as cohomology groups. We have $\overline{N}^\cdot = \mathrm{Hom}_R(F_\cdot, \overline{N}) = \mathrm{Hom}_{\overline{R}}(\overline{F}_\cdot, \overline{N})$ and as M is S - flat, $\overline{M}$ is resolved by the complex $\overline{F}_\cdot$. Hence, the cohomology of the complex $\overline{N}^\cdot$ computes $\mathrm{Ext}^i_{\overline{R}}(\overline{M},\overline{N})$. Consider now the functor T^i on S - modules:

$$T^i : A \longmapsto T^i(A) := H^i(N^\cdot \otimes_S A).$$

By the usual arguments one has:

* T^i left exact $\Leftrightarrow W^i := \mathrm{Coker}(N^{i-1} \longrightarrow N^i)$ is S - flat.

* T^i right exact $\Leftrightarrow \varphi_i : T^i(S)\otimes_S A \twoheadrightarrow T^i(A)$, for all A

  $\Leftrightarrow \varphi_i$ is an isomorphism for all A.

But also: T^i right exact $\Leftrightarrow T^{i+1}$ left exact $\Leftrightarrow W^{i+1}$ S - flat $\Leftrightarrow$ (local criterion for flatness, [Ma], pp. 145-149) $W^{i+1}\otimes m_S \hookrightarrow W^{i+1}$ $\Leftrightarrow T^{i+1}(m_S) \hookrightarrow T^{i+1}(S) \Leftrightarrow T^i(S)\otimes k \twoheadrightarrow T^i(k)$. From this the proposition follows. ⊠

Corollary (0.2) :

Under the same assumptions as in proposition (0.1) one has :

i) If $\mathrm{Ext}^i_{\overline{R}}(\overline{M},\overline{N}) = 0$ then $\mathrm{Ext}^i_R(M,N) = 0$.

ii) If $\mathrm{Ext}^k_{\overline{R}}(\overline{M},\overline{N}) = 0$ for $k = i-1$ and $k = i+1$, then $\mathrm{Ext}^i_R(M,N)$ is S-flat and $\mathrm{Ext}^i_{\overline{R}}(\overline{M},\overline{N}) = \mathrm{Ext}^i_R(M,N)\otimes_S k$.

proof: Statement i) follows easily from (0.1) together with the lemma of Nakayama, as we know that the modules $\mathrm{Ext}^i_R(M,N)$ are finitely generated modules over R. For statement ii) note as φ_{i+1} is surjective, the functor T^{i+1} is right exact and as $T^{i+1}(S) = 0$ by i), we find that $T^{i+1}(m_S) = 0$ and hence φ_i is surjective, so by (0.1) ii) we are done. ⊠

Lemma (0.3) :

Let M be any finitely generated R - module and N be an S - flat R - module.

Then : $Ext^i_{\bar{R}}(\bar{M},\bar{N}) = 0$ for $i = 0, 1, \ldots, p$ implies

$Ext^i_R(M,N) = 0$ for $i = 0, 1, \ldots, p$.

proof: By [Ma], thm 28, p. 100 we have that $Ext^i_{\bar{R}}(\bar{M},\bar{N}) = 0$ for $i = 0, 1, \ldots, p$ is equivalent to the existence of elements $\bar{x}_i \in \bar{R}$, $i = 0, 1, \ldots, p$ such that

i) $\bar{x}_i \in Ann_{\bar{R}}(\bar{M})$

ii) the $\bar{x}_i$ form a regular $\bar{N}$ - sequence.

Now let $m_1, m_2, \ldots, m_t$ be R - generators for M and $x \in R$ any lift of one of the $\bar{x}_i$. Then $x.M \subset m_S .M$, so $\det(x.I - B).M = 0$, (where B is any matrix of x. with respect to the generators m_i of M) by Cramers rule. As the entries of the matrix B are in the maximal ideal, we see that the elements $y_i := \det(x_i .I - B) \in Ann_R(M)$ project to $\bar{x}_i^t$. As these form a regular $\bar{N}$ - sequence and N is S- flat, we have that the y_i form a regular N - sequence (see [Ma], pp. 150-151). Hence the lemma follows by application of [Ma], thm. 28 again. ⊠

Definition (0.4) :

Let R and S as above and let M be an R - module. We say that:

* M is *Cohen - Macaulay over S* (CM over S) if and only if

i) $\bar{M}$ is a Cohen - Macaulay $\bar{R}$ - module (i.e. $\dim_{\bar{R}}(\bar{M}) = \operatorname{depth}_{\bar{R}}(\bar{M})$).

ii) M is S - flat.

We call $d := \dim_{\bar{R}}(\bar{M})$ the *dimension* and $c := \dim(\bar{R}) - d$ the *codimension* of M over S. If $c = 0$ we say that M is maximal Cohen-Macaulay over S (M is MCM over S).

* R is *regular over S* if and only if

i) $\bar{R}$ is a regular local ring.

ii) R is S - flat.

We call $N := \dim(\bar{R})$ the *relative dimension* of R over S.

For a local ring that is regular over S we will use the symbol P. We call $\omega_{P/S} := P$ the dualizing module of P over S.

Proposition (0.5) :

Let P be regular over S of relative dimension N. For an S - flat P - module M the following conditions are equivalent:

i) M is CM over S of codimension c.

ii) $\mathrm{Ext}^i_{\bar{P}}(\bar{M}, \omega_{\bar{P}}) = 0$ for $i \neq c$.

proof : First assume i). The relation between depth and local cohomology (see [Gro], cor. 3.10, p.47) tells us that $H^i_m(\bar{M})=0$ for $i < N - c$. Then the local duality theorem for the regular local ring $\bar{P}$ (see [Gro], thm 6.3, p.85) states that $H^i_m(\bar{M})$ is (Matlis-)dual to $\mathrm{Ext}^{N-i}_{\bar{P}}(\bar{M}, \omega_{\bar{P}})$. Hence we have $\mathrm{Ext}^k_{\bar{P}}(\bar{M}, \omega_{\bar{P}}) = 0$ for $k > c$. The vanishing of the lower Ext's follows by Ischebek's lemma ([Ma], (15.E), p.104), because the dimension of $\bar{M}$ is $N-c$ and the depth of $\omega_{\bar{P}}$ is N. Hence we get ii). To get i) from ii) one just reverses the above steps. ⊠

Definition (0.6) :

Let P be regular over S and let M be a P -module which is CM over S of codimension c. The *dual module* of M is defined to be

$$M^{\vee} := \mathrm{Ext}^c_P(M, \omega_{P/S}) .$$

An S - algebra R is called *embeddable* if R is the quotient of a ring P which is regular over S. If R is Cohen - Macaulay over S of codimension c considered as a P - module, we define the *dualizing module* to be $\omega_{R/S} := R^{\vee} = \mathrm{Ext}^c_P(R, \omega_{P/S})$.

Proposition (0.7) :

Let P be regular over S and let M be CM over S of codimension c. Then one has:

i) $\mathrm{Ext}_P^k(M,\omega_{P/S}) = 0$ for $k \neq c$.

ii) the dual module $M^\vee$ is S - flat .

iii) $\overline{(M^\vee)} = (\overline{M})^\vee$.

proof : Combine (0.5) with (0.2). In fact, for an S - flat module M, the Cohen - Macaulay property is equivalent to the above three properties. ⊠

Remark (0.8) :

By the change-of-rings spectral sequence (see [C-E], p. 349)

$$E_2^{p\,q} = \mathrm{Ext}_R^p(M,\mathrm{Ext}_P^q(R,N)) \Rightarrow \mathrm{Ext}_P^{p+q}(M,N)$$

one can relate Ext's over different rings. If R is embeddable and CM over S of codimension c as a P - module then one has an isomorphism

$$\mathrm{Ext}_R^p(M,\omega_{R/S}) = \mathrm{Ext}_P^{p+c}(M,\omega_{P/S})$$

for any R - module M. This also shows that $\omega_{R/S}$ is essentially independent of the choice of P in (0.6).

Corollary (0.9) :

Let R be embeddable and CM over S of codimension c. Then one has:

i) Propositions (0.5) and (0.7) hold for P replaced by R.

ii) If M is CM over S of codimension e considered as an R - module, then M is CM over S of codimension e+c considered as a P - module.

Proposition (0.10) :

Let R be embeddable and CM over S and let M be an R - module which is CM over S of codimension c. Then one has:

i) $M^\vee = \mathrm{Ext}^c_R(M, \omega_{R/S})$ is also CM over S of codimension c.

ii) There is a natural isomorphism $M \longrightarrow (M^\vee)^\vee$.

proof : It is not hard to see that by using (0.9) one can reduce to the case that R = P , P regular over S. Consider a minimal free resolution $F_. \longrightarrow M$ of M over P. Because M is S - flat, the complex $\overline{F}_.$ is a minimal free resolution for $\overline{M}$. Because $\overline{M}$ is Cohen - Macaulay of codimension c over the regular local ring $\overline{P}$, we conclude by the Auslander -Buchsbaum formula (see [A-B], thm 2.3, p.397) that the length of the complex $F_.$ is exactly c, i.e. the resolution looks like

$$0 \longrightarrow F_c \longrightarrow F_{c-1} \longrightarrow \dots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0.$$

When we apply $\mathrm{Hom}_P(-, P)$ we get the complex

$$0 \longrightarrow F_0^\vee \longrightarrow F_1^\vee \longrightarrow \dots \longrightarrow F_{c-1}^\vee \longrightarrow F_c^\vee \longrightarrow M^\vee \longrightarrow 0$$

where $F^\vee_. = \mathrm{Hom}_P(F_., P)$. By (0.7) this last complex is exact, hence we get a free resolution of $M^\vee$. As we already know that $M^\vee$ is S - flat by (0.7) one can reverse the steps and the result follows. ⊠

§ 1. *The Functor of Admissible Deformations.*

Consider a (germ of a) space X and a subspace Σ contained in the singular locus of X. In this paragraph we define a certain subfunctor of the functor of deformations of the inclusion map of Σ in X, which consists of deformations for which the deformed Σ stays inside the singular locus of the deformed X. First we have to define an appropriate structure on the critical locus of a map (and hence on the singular locus of a space). Let $X \longrightarrow S$ be a flat mapping of relative dimension n.

Definition (1.1) :

The *critical locus* $\mathcal{C} := \mathcal{C}_{X/S}$ is the locus defined by $F_n(\Omega^1_{X/S})$, the n-th Fitting ideal of the sheaf of relative Kähler one-forms. The *critical space* is $\mathcal{C}$ together with $\mathcal{O}_{\mathcal{C}} := \mathcal{O}_X/F_n(\Omega^1_{X/S})$ as structure sheaf.

This definition can be found in [Te], def.2.5, p.587. It is natural to consider the critical space again as a space over S. One of the reasons to define the critical space in this way is because of the following

Property (1.2) :

The formation of the critical space commutes with base-change. This comes down to a simple property of Fitting ideals (see [Te], p.570)

Definition (1.3) :

* A *diagram over S* is a triple (Σ_S , X_S , i), where Σ_S and X_S are spaces over S and $i : \Sigma_S \longrightarrow X_S$ is a map over S. Usually we will be sloppy and say that $\Sigma_S \longrightarrow X_S$ is a diagram over S, without even mentioning the map.

* A *morphism of diagrams* is defined in the obvious way.

* A diagram $\Sigma_S \longrightarrow X_S$ over S is said to be *admissible*, if the map $i: \Sigma_S \longrightarrow X_S$ factorizes over the inclusion map $\mathcal{C}_{X_S/S} \hookrightarrow X_S$.

* A morphism between admissible diagrams over S is just a morphism of the underlying diagrams over S.

* Let $\Sigma \longrightarrow X$ be a diagram over Spec(k), k a field. Let S be the spectrum of a local ring with residue field k. A diagram $\Sigma_S \longrightarrow X_S$ over S is said to be a *deformation* of the diagram $\Sigma \longrightarrow X$ iff :

i) Σ_S and X_S are flat over S.

ii) $\left(\Sigma \longrightarrow X\right) \approx \left(\Sigma_S \longrightarrow X_S\right) \underset{S}{\times} \mathrm{Spec}(k)$

* A deformation $\Sigma_S \longrightarrow X_S$ of $\Sigma \longrightarrow X$ is called *admissible* or is said to be an *admissible deformation* if the diagram $\Sigma_S \longrightarrow X_S$ is admissible.

Let $\mathbf{C}$ denote the category of local noetherian k - algebras with residue field k. It has a full subcategory $\mathbf{C}_a$ consisting of Artinian algebras. Let **Set** denote the category of sets.

Definition (1.4) :

Let $\Sigma \longrightarrow X$ be an admissible diagram over Spec(k).

The functor $\mathbf{C} \longrightarrow$ **Set**

$$S \longmapsto \{\text{isomorphism classes of deformations of } \Sigma \longrightarrow X \text{ over } S\}$$

is called the *functor of deformations of the diagram* $\Sigma \longrightarrow X$ and is denoted by $\mathrm{Def}(\Sigma \longrightarrow X)$.

The functor $\mathbf{C} \longrightarrow$ **Set**

$$S \longmapsto \{\text{isomorphism classes of admissible deformations of } \Sigma \longrightarrow X \text{ over } S\}$$

is called the *functor of admissible deformations* and is denoted by $\mathrm{Def}(\Sigma, X)$.

We remark that the base-change property (1.2) is needed to make Def(Σ,X) into a functor. Remark further that Def(Σ,X) is a subfunctor of Def(Σ ⟶ X). We do not make a notational distinction between these functors and their restriction to the subcategory $\mathbf{C_a}$.

We recall that if T' and T $\in$ Ob($\mathbf{C_a}$) then α: T' ⟶ T is called a *simple surjection* if α is a surjection and Ker(α) is a principal ideal in T' with $\mathrm{Ker}(\alpha).m_{T'} = 0$, where $m_{T'}$ is the maximal ideal of T'. (see [Schl 1], 1.2).

Lemma (1.5):

The functor F := Def(Σ ⟶X) is semi-homogeneous, i.e. three of the four *Schlessinger conditions* are satisfied:

i) $F(k) = \{pt\}$

ii) $F(T''\times_T T') \longrightarrow F(T'')\times_{F(T)} F(T')$ is surjective for every simple surjection T' ⟶ T and every morphism T'' ⟶ T.

iii) $F(T'\times_k k[\varepsilon]/\varepsilon^2) \longrightarrow F(T')\times F(k[\varepsilon]/\varepsilon^2)$ is an isomorphism for all T'.

The proof is similar to [Schl1], 3.7 . In fact, for ii), if we are given deformations $\Sigma_S \longrightarrow X_S$, $\Sigma_{S'} \longrightarrow X_{S'}$ and $\Sigma_{S''} \longrightarrow X_{S''}$ (S = Spec(T) etc.) with

$$\left(\Sigma_{S'} \longrightarrow X_{S'}\right)\times_{S'} S \approx \left(\Sigma_{S''} \longrightarrow X_{S''}\right)\times_{S''} S \approx \left(\Sigma_S \longrightarrow X_S\right)$$

then the natural map

$$\left(\left(\Sigma_{S''} \amalg_{\Sigma_S} \Sigma_{S'}\right) \longrightarrow \left(X_{S''} \amalg_{X_S} X_{S'}\right)\right) \qquad (*)$$

gives a deformation of the diagram over $S'\times_S S''$ which restricts to the given deformations over S' and S''.

Proposition (1.6):

Def(Σ,X) is a semi-homogeneous subfunctor of Def(Σ ⟶ X).

proof: Our definitions are casted in such a way that the proof just becomes a repetition of the proof that the functor of *deformations with a singular section* has an analogous property. That case corresponds to $\Sigma = \{pt\}$ and has been treated by Buchweitz (see [Bu], p.79). We keep the notation as above, but now we are given $\Sigma_S \longrightarrow X_S$, etc. , which are admissible. We have to show that the diagram (*) under (1.5) in fact is admissible. It is clear that the map (*) factorizes over

$$\mathcal{C}_{X_{S''}/S''} \amalg_{\mathcal{C}_{X_S/S}} \mathcal{C}_{X_{S'}/S'}$$

But by the base change property of the critical locus (1.2) there is a natural morphism

$$\mathcal{C}_{X_{S''}/S''} \amalg_{\mathcal{C}_{X_S/S}} \mathcal{C}_{X_{S'}/S'} \longrightarrow \mathcal{C}_{X_{S''} \amalg_{X_S} X_{S'} / S'' \times_S S'}$$

which gives us the factorization which shows the admissibility of (*). It is now clear that the result follows because $\mathrm{Def}(\Sigma \longrightarrow X)$ itself is a semi-homogeneous functor. ⊠

Corollary (1.7) :

If $T^1(\Sigma,X) := \mathrm{Def}(\Sigma,X)(k[\varepsilon]/(\varepsilon^2))$ is a finite dimensional vector space, then $\mathrm{Def}(\Sigma,X)$ satisfies the Schlessinger conditions; i.e. $\mathrm{Def}(\Sigma,X)$ has a *hull* (i.e. a there is a 'formal' semi-universal deformation).

proof : See [Schl 1], 2.11 . ⊠

Suppose that we have an admissible diagram $\Sigma \hookrightarrow X$ and an embedding of X in some smooth ambient space Y. Analogous to the functor $\mathrm{Def}(\Sigma,X)$ of admissible deformations one can define a functor $\mathrm{Embdef}(\Sigma,X)$ of admissible deformations which can be realized inside Y. It is of some importance to describe the relation between $\mathrm{Def}(\Sigma,X)$ and $\mathrm{Embdef}(\Sigma,X)$, because in practice one always describes X and Σ by *equations*, so an embedding is always implicit. We now shall make this relation more precice.

Definition (1.8) :

Let Y be a space smooth over k. An *embedded admissible diagram* (over Spec(k)) is a diagram $\Sigma \hookrightarrow X \hookrightarrow Y$ (over Spec(k)) such that $\Sigma \hookrightarrow X$ is admissible. An *embedded admissible deformation* over S is a diagram $\Sigma_S \hookrightarrow X_S \hookrightarrow Y_S \approx Y \times S$ over S such that $\Sigma_S \hookrightarrow X_S$ is an admissible deformation of $\Sigma \hookrightarrow X$. Morphisms between such objects are defined in the obvious way. The functor

$$\begin{aligned} \mathbf{C} &\longrightarrow \mathbf{Set} \\ S &\longmapsto \left\{ \begin{array}{l} \text{isomorphism classes of embedded admissible} \\ \text{deformations of } \Sigma \hookrightarrow X \hookrightarrow Y \text{ over } S \end{array} \right\} \end{aligned}$$

is called the *functor of embedded admissible deformations* and is denoted by Embdef(Σ,X), the space Y being understood.

Lemma (1.9) :

The natural forgetful transformation

$$\mathrm{Embdef}(\Sigma,X) \longrightarrow \mathrm{Def}(\Sigma,X)$$

is smooth.

This statement is completely analogous to the corresponding statement about ordinary deformations. We omit the proof and refer to [Ar1] for further details.

§ 2. *Injectivity.*

In § 1. we introduced the functor Def(Σ,X) of admissible deformations, consisting of deformations of X together with a subspace Σ of the critical space $\mathcal{C}$ of X. There is a natural forgetful transformation

$$\mathrm{Def}(\Sigma, X) \longrightarrow \mathrm{Def}(X) .$$

In this section we formulate some conditions under which this is an injection, i.e. conditions under which $\mathrm{Def}(\Sigma,X)(S) \hookrightarrow \mathrm{Def}(X)(S)$ is injective for all S in the category $\mathbf{C}$. Intuitively, this should be the case when Σ can not 'move' inside $\mathcal{C}$. One expects this to be the case when Σ and $\mathcal{C}$ are equal at the generic points. The problem is to find a good functorial way to reconstruct Σ from X alone. The conditions we find are probably unnecessarily strong, but they suffice for the applications we have in mind. We use some generalities from local algebra which can be found in § 0.

Lemma (2.1):
Let R and S be rings as in § 0. Consider an exact sequence of R - modules:

$$0 \longrightarrow N \longrightarrow A \longrightarrow M \longrightarrow 0$$

Assume that:

i) $\overline{M}$ is Cohen-Macaulay of codimension c.

ii) M is S - flat.

iii) $\mathrm{Ext}^i_{\bar{R}}(\overline{N}, \omega_{\bar{R}}) = 0$ for i = 0, 1, . . . , c.

Then $M \approx (A^\vee)^\vee := \mathrm{Ext}^c_R(\mathrm{Ext}^c_R(A, \omega_{R/S}), \omega_{R/S})$.

proof : By (0.4) M is CM over S of codimension c. By (0.3) we have that $\mathrm{Ext}^i_R(N, \omega_{R/S}) = 0$ for i = 0, 1, . . . , c. When we take $\mathrm{Hom}_R(- , \omega_{R/S})$ of the above exact sequence we get $M^\vee \approx A^\vee$. Hence the lemma follows from (0.10). ⊠

The above lemma expresses the fact that if the 'difference' $\overline{N}$ between $\overline{A}$ and $\overline{M}$ is 'small', then a possible flat deformation of $\overline{M}$ to M is completely determined by A, even if A is not flat. For this to be true, one of course needs some purity of M, like the CM - assumption. We use this fact to prove the following theorem:

Proposition (2.2) :

Let $\Sigma \hookrightarrow X$ an admissible diagram over Spec(k) and let I be the ideal of Σ in $\mathcal{O}_X$. Assume that:

i) X is Cohen-Macaulay of dimension n.

ii) Σ is Cohen-Macaulay of codimension c in X.

iii) $\mathcal{E}xt^i_X(I/F_n(\Omega^1_X), \omega_X) = 0$ for i = 0, 1, . . . , c.

Then the natural forgetful transformation

$$\mathrm{Def}(\Sigma, X) \longrightarrow \mathrm{Def}(X)$$

is injective.

proof : Let $\Sigma_S \hookrightarrow X_S$ be an admissible deformation of $\Sigma \hookrightarrow X$ over S. Because $\Sigma_S \hookrightarrow \mathcal{C}_S$, where $\mathcal{C}_S$ is the critical space of X_S over S, we get an exact sequence of $\mathcal{O}_{X_S}$ - modules:

$$0 \longrightarrow \mathcal{N} \longrightarrow \mathcal{O}_{\mathcal{C}_S} \longrightarrow \mathcal{O}_{\Sigma_S} \longrightarrow 0$$

where $\mathcal{N} = I_S/F_n(\Omega^1_{X_S/S})$ (I_S the ideal of Σ_S in $\mathcal{O}_{X_S}$).

Our assumptions are of such a nature that we can apply lemma (2.1) to get $\mathcal{O}_{\Sigma_S} \approx (\mathcal{O}_{\mathcal{C}_S}{}^{\vee})^{\vee}$. Hence the arrow $\mathcal{O}_{X_S} \longrightarrow \mathcal{O}_{\Sigma_S}$ is naturally identified with the composition $\mathcal{O}_{X_S} \longrightarrow \mathcal{O}_{\mathcal{C}_S} \longrightarrow (\mathcal{O}_{\mathcal{C}_S}{}^{\vee})^{\vee}$. As the critical space $\mathcal{C}_S$ is determined in a canonical way by $X_S \longrightarrow S$, we see that we can reconstruct $\Sigma_S \hookrightarrow X_S$ from the map $X_S \longrightarrow S$ alone; i.e. $Def(\Sigma, X) \longrightarrow Def(X)$ is injective. ⊠

Corollary (2.3) :

Let $\Sigma \hookrightarrow X$ be an admissible deformation over Spec(k). Assume that Σ and X are Cohen-Macaulay. If $\dim\left(Supp(I/F_n(\Omega^1_X))\right) < \dim(\Sigma)$, then the transformation $Def(\Sigma,X) \longrightarrow Def(X)$ is injective.

proof : This follows immediately from (2.2), because by local duality iii) is equivalent to $H^i_{\{0\}}(I/F_n(\Omega^1_X)) = 0$ for $i \geq \dim(\Sigma)$. As local cohomology of a module vanishes above the dimension of its support, we get the result. ⊠

In the case that X is a (germ of a) reduced hypersurface singularity, given by an equation of the form $f=0$, $f \in \mathbb{C}\{x_0, x_1, \ldots, x_n\}$, then

$$I/F_n(\Omega^1_X) = I/(f,J(f))$$

where $J(f) := (\partial_0 f, \partial_1 f, \ldots, \partial_n f)$ is the Jacobian ideal, generated by the partial derivatives $\partial_i f = \partial f/\partial x_i$. A further specialisation of (2.3) is the following.

Corollary (2.4) :

Let X be a hypersurface germ defined by $f \in \mathbb{C}\{x_0, x_1, \ldots, x_n\}$ and let Σ be defined by an ideal $I \supset (f,J(f))$. Assume that:

i) Σ is Cohen-Macaulay of dimension ≥ 1

ii) $\dim_{\mathbb{C}}(I/(f,J(f)) < \infty$.

Then the forgetful transformation $Def(\Sigma, X) \longrightarrow Def(X)$ is injective.

Remark (2.5) :

When the ideal I is *reduced* and the conditions of (2.4) apply, we say that X has (generically) transverse A_1 - singularities. In the above context of hypersurface singularities, Pellikaan [Pe] studied modules of the form I/J(f). His results imply the following:

*Theorem (*Pellikaan, [Pe 4], thm. (3.3), (3.4), (3.5)*)*
Let $f \in \mathbb{C}\{x_0, x_1, \ldots, x_n\}$ define a germ of a mapping $f: \mathbb{C}^{n+1} \longrightarrow \mathbb{C}$. Let Σ be defined by an ideal $I \supset J(f)$ such that $\dim_{\mathbb{C}}(I/J(f)) < \infty$. Assume that one of the following conditions hold:

i) Σ is Cohen-Macaulay and $\dim(\Sigma) = 1$.

ii) Σ is a complete intersection.

iii) Σ is Cohen-Macaulay of codimension 2.

Then for an admissable deformation of the mapping f and Σ (defined analoguous to (1.3)) I/J(f) is *flat.*

Note however that the module I/(f,J(f)) can *not* expected to behave in a flat way. (c.f. μ and τ for an isolated hypersurface singularity.)

We conclude this section with an example that will also play a role in § 4. and which shows that $\mathrm{Def}(\Sigma,X) \longrightarrow \mathrm{Def}(X)$ is not always injective.

Example (2.6) :

Let X be defined by $f = x^3 + y^2 \in k[x,y]$ (char(k) $\neq$ 2,3), so $J_f = (x^2, y)$. Let Σ be the subspace of the critical locus defined by I = (x,y). Consider the trivial deformation of X over $k[\varepsilon]/(\varepsilon^2)$, defined by the same function f, but now considered in $k[\varepsilon, x, y]/(\varepsilon^2)$. Let $I_1 = (x,y) \subset k[\varepsilon,x,y]/(\varepsilon^2)$ and $I_2 = (x+\varepsilon, y) \subset k[\varepsilon,x,y]/(\varepsilon^2)$. Because $x^2 = (x+\varepsilon).(x-\varepsilon)$ we see that both I_1 and I_2 correspond to admissible deformations of the pair Σ, X. One can check that these elements are different in $\mathrm{Def}(\Sigma, X)(k[\varepsilon]/(\varepsilon^2))$, but map to the trivial deformation of X in $\mathrm{Def}(X)(k[\varepsilon]/(\varepsilon^2))$. Hence $\mathrm{Def}(\Sigma, X) \longrightarrow \mathrm{Def}(X)$ is not injective in this example.

§ 3. Infinitesimal Admissible Deformations of Hypersurfaces

This paragraph is devoted to the study of the functor $\mathrm{Def}(\Sigma,X)$ in the case that X is a germ of a hypersurface singularity. This study is divided into three parts. In part A we study the vector space $T^1(\Sigma,X) := \mathrm{Def}(\Sigma,X)(k[\varepsilon]/(\varepsilon^2)$ of first order admissible deformations. Part B is devoted to the obstruction theory, i.e. conditions for extending a given deformation over a somewhat bigger base space. Under appropriate circumstances the obstruction space we get is a quotient of (the torsion subsheaf of) Ω^1_Σ , the Kähler differentials on Σ. We also prove a theorem that states roughly that the base space of the semi universal deformation space of $\mathrm{Def}(\Sigma,X)$ depends *mainly* on Σ. In part C we prove that if Σ is not obstructed, then the obstruction space is in fact an in general much smaller subspace of the quotient of $\mathrm{Tors}(\Omega^1_\Sigma)$ we got in B. A crucial rôle is played by the so called Hessian form. For computational purposes this does not seem to be to important, but for theorètical purposes it probably is. Further we give some formulas relating this Hessian to other invariants.

Notations and Conventions (3.1):

Throughout this paragraph X will denote a germ of a hypersurface in $(\mathbb{C}^{n+1},0)$, defined by an equation $f = 0$, $f \in \mathcal{O} := \mathcal{O}_{\mathbb{C}^{n+1},0} = \mathbb{C}\{x_0, x_1, \ldots, x_n\}$. In fact, as all our arguments will be algebraic in nature, we might as well replace $\mathcal{O}$ by any local ring which is smooth over a field k.

Σ will be a subspace of X, defined by an ideal $I \subset \mathcal{O}$.

Furthermore, we put

$$\textstyle\int I := \left\{ g \in \mathcal{O} \,\middle|\, (g, \partial_i g) \subset I \right\} \tag{1}$$

where $\partial_i g := \partial g / \partial x_i$ is the partial derivative of g with respect to x_i. $\int I$ is an ideal and is called the *primitive ideal* of I (see [Pe3], def. 1.1) This notion of primitive ideal leads to a convenient formulation of the condition that Σ is contained in the critical locus of X. Clearly:

$$\Sigma \subset \mathfrak{C}_X \quad \Leftrightarrow \quad f \in \int I$$

An alternative way to define $\int I$ is by the exact sequence

$$0 \longrightarrow I/\int I \xrightarrow{d} \Omega^1 \otimes \mathcal{O}_\Sigma \longrightarrow \Omega^1_\Sigma \longrightarrow 0 \qquad (2)$$

where $\Omega^1 := \Omega^1_{\mathbb{C}^{n+1},0}$

We choose generators Δ_i , $i = 1, 2, \ldots, m$ for the ideal I:

$$I = \left(\Delta_1, \Delta_2, \ldots, \Delta_m \right) \qquad (3)$$

Because $f \in I\ (\Sigma \subset X)$ we can write:

$$f = \sum_{i=1}^{m} \alpha_i \cdot \Delta_i \qquad (4)$$

Because $f \in \int I$, there are elements $\omega_i \in \Omega^1$, $i = 1, 2, \ldots, m$ such that

$$df = \sum_{i=1}^{m} \omega_i \cdot \Delta_i \qquad (5)$$

In order to keep the notation as simple as possible we suppress all indices, i.e. we simply write $I = (\Delta)$, $f = \alpha.\Delta$ and $df = \omega.\Delta$ instead of (3), (4) and (5). We will extend this 'summation convention' without any further comment to new situations. To make this more precise, we choose a presentation of $\mathcal{O}_\Sigma$ as an $\mathcal{O}$ - module

$$0 \longrightarrow \mathcal{R} \longrightarrow \mathcal{F} \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}_\Sigma \longrightarrow 0 \qquad (6)$$

where $\mathcal{F} := \mathcal{O}^m$ and $\mathcal{F} \longrightarrow \mathcal{O}$ is the map $e_i \longmapsto \Delta_i$. $\mathcal{R}$ is the module of *relations* between the generators Δ_i , i.e. $r \in \mathcal{R} \Leftrightarrow r.\Delta = 0$. Then α can be considered as an element of $\mathcal{F}$ and is determined by f modulo $\mathcal{R}$ and ω can be considered as an element of $\mathcal{F} \otimes \Omega^1$ and is determined modulo $\mathcal{R} \otimes \Omega^1$.

So we have: $f \in \int I \Leftrightarrow \exists\ \alpha, \omega \mid f = \alpha.\Delta$ and $df = \omega.\Delta$. We will use frequently the following equivalent form of this statement.

Lemma (3.2): $f \in \int I \ \Leftrightarrow\ \exists\ \alpha, \Gamma \mid f = \alpha.\Delta$ and $\alpha.d\Delta + \Gamma.\Delta = 0$.

proof : From $f = \alpha.\Delta$ we get $df = d\alpha.\Delta + \alpha.d\Delta$. As $df = \omega.\Delta$ we get $0 = \alpha.d\Delta + \Gamma.\Delta$, where $\Gamma = d\alpha - \omega$. ⊠

$\mathcal{A}$. *First Order Deformations and* $T^1(\Sigma,X)$.

Let Embdef(Σ,X) be the functor of admissible embedded deformations of Σ in X. There are two obvious forgetful transformations

$$P_X : \text{Embdef}(\Sigma,X) \longrightarrow \text{Embdef}(X)$$
$$P_\Sigma : \text{Embdef}(\Sigma,X) \longrightarrow \text{Embdef}(\Sigma)$$

We put
$$\mathcal{A} := \text{Embdef}(\Sigma,X)(S)$$
$$N_X := \text{Embdef}(X)(S)$$
$$N_\Sigma := \text{Embdef}(\Sigma)(S)$$

where $S = \mathbb{C}[\varepsilon]/(\varepsilon^2)$. Via the natural mappings $P_X : \mathcal{A} \longrightarrow N_X$ and $P_\Sigma : \mathcal{A} \longrightarrow N_\Sigma$ we can consider $\mathcal{A}$ as a subset of $N_\Sigma \times N_X$. We call $\mathcal{A}$ the set of *admissible pairs.* The set $P_\Sigma(\mathcal{A})$ we call the set of admissible normal vectors to Σ and $P_X(\mathcal{A})$ the set of admissible functions. All these sets have a natural $\mathbb{C}$-vector space structure.

Let us first briefly recall the description of the vector spaces $N_\Sigma := \text{Embdef}(\Sigma)(\mathbb{C}[\varepsilon]/(\varepsilon^2))$ and $T^1_\Sigma := \text{Def}(\Sigma)(\mathbb{C}[\varepsilon]/(\varepsilon^2))$ of first order (embedded) deformations (see also [Ar 1]). Let $\Sigma \subset \mathbb{C}^{n+1}$ be defined by an ideal $I = (\Delta) \subset \mathcal{O}$ and consider a flat deformation Σ_ε of Σ over Spec$(\mathbb{C}[\varepsilon]/(\varepsilon^2))$. Then Σ_ε is defined by an ideal $I_\varepsilon = (\Delta_\varepsilon) = (\Delta + \varepsilon.\, n)$ $\subset \mathcal{O}[\varepsilon]/(\varepsilon^2)$. Now flatness in terms of relations means that for all relations $r \in \mathcal{R}$ for (Δ) there is a relation $r_\varepsilon = r + \varepsilon.s$ for (Δ_ε), i.e. $r_\varepsilon \cdot \Delta_\varepsilon = 0$. Expanding the product and using $\varepsilon^2 = 0$ we see that an embedded first order deformation is given by an n such that for all $r \in \mathcal{R}$ one has $r.\,n + s.\Delta = 0$ for some s. So n can be considered as a homomorphism $I \longrightarrow \mathcal{O}_\Sigma$; $\Delta_i \longmapsto n_i$. (In the sequel we usually will not make a distinction between a normal vector n and its set of components (n_i)). From this it follows that

$$N_\Sigma = \text{Hom}(I,\mathcal{O}_\Sigma) = \text{Hom}_\Sigma(I/I^2,\mathcal{O}_\Sigma) \qquad (7)$$

The $\mathcal{O}_\Sigma$ - module on the right hand side is called the *normal sheaf* (of Σ in the ambient space), being the dual of the conormal sheaf I/I^2.

The space of first order deformations is obtained from the space of first order embedded deformations by dividing out by the infinitesimal automorphisms. These are generated by the vector fields Θ on the ambient space: a $\vartheta \in \Theta$ gives rise to the homomorphism $\left(\Delta_i \longmapsto \vartheta(\Delta_i)\right) \in N_\Sigma$. Thus one sees that T^1_Σ sits in an exact sequence

$$0 \longrightarrow \Theta_\Sigma \longrightarrow \Theta \otimes \mathcal{O}_\Sigma \longrightarrow N_\Sigma \longrightarrow T^1_\Sigma \longrightarrow 0 \qquad (8)$$

which starts with the dual of the sequence (2).
The description of N_X and T^1_X for the hypersurface X is of course very easy: $N_X \approx \mathcal{O}_X = \mathcal{O}/(f)$ and $T^1_X \approx \mathcal{O}/(f, \partial_i f) = \mathcal{O}_{\mathcal{C}_X}$.

Now we can describe the set $\mathcal{A}$ of admissible pairs as follows:

$$\mathcal{A} = \Big\{ (n, g) \in N_\Sigma \times N_X \,\Big|\, \exists\, \alpha_1 \in \mathcal{F},\ \omega_1 \in \mathcal{F} \otimes \Omega^1 \text{ such that}$$
$$1)\quad (f + \varepsilon.g) = (\alpha + \varepsilon.\alpha_1).(\Delta + \varepsilon.n)$$
$$2)\ d(f + \varepsilon.g) = (\omega + \varepsilon.\omega_1).(\Delta + \varepsilon.n) \Big\}$$

These conditions can be rewritten as

$$\exists\, \alpha_1 \in \mathcal{F},\ \omega_1 \in \mathcal{F} \otimes \Omega^1 \,\Big|\, \begin{array}{l} 1)\quad g = \alpha.n + \alpha_1.\Delta \\ 2)\ d\,g = \omega.n + \omega_1.\Delta \end{array} \qquad (9)$$

These expressions motivate the following definitions.

Definition (3.3) :

The α and the ω - map are given by:

$$\alpha_f: N_\Sigma \longrightarrow \mathcal{O}_\Sigma \quad ;\ n \longmapsto \alpha.n$$
$$\omega_f: N_\Sigma \longrightarrow \Omega^1 \otimes \mathcal{O}_\Sigma \ ;\ n \longmapsto \omega.n$$

Furthermore we define the w - map by

$$w_f : N_\Sigma \longrightarrow \Omega^1_\Sigma \quad ;\ n \longmapsto d(\alpha.n) - \omega.n = \alpha.dn + \Gamma.n$$

We usually omit the index f if no confusion is likely.

Lemma (3.4) :

The α, ω and w - map only depend on $f \in \int I$ and not on the particular choices of the α_i and ω_i.

proof : The α-map can be defined intrinsically as the evalution of the normal vector n, considered as a homomorphism, on f. Similarly, the ω-map is intrinsically $n \longmapsto (n \otimes Id)(df)$, and the result follows. ⊠

Note that the α and the ω - map are homomorphisms of $\mathcal{O}_\Sigma$ - modules, but in general w is only $\mathbb{C}$-linear.

Lemma (3.5) :

i) There are exact sequences

$$0 \longrightarrow P_\Sigma(\mathcal{A}) \longrightarrow N_\Sigma \xrightarrow{\ w\ } \Omega^1_\Sigma$$
$$0 \longrightarrow \int I/(f) \longrightarrow \mathcal{A} \longrightarrow P_\Sigma(\mathcal{A}) \longrightarrow 0$$

ii) There is a natural map $\varphi : P_\Sigma(\mathcal{A}) \longrightarrow \mathcal{O}/\int I$ which associates to an admissible normal vector $n \in P_\Sigma(\mathcal{A})$ a unique $\int I$ - coset such that

$$(n,g) \in \mathcal{A} \Leftrightarrow g \in \varphi(n) + \int I .$$

proof : One has: $n \in P_\Sigma(\mathcal{A}) \Leftrightarrow \exists g$ such that $(n,g) \in \mathcal{A} \Leftrightarrow \exists\ \alpha_1, \omega_1$ such that i) $g = \alpha.n + \alpha_1.\Delta$ and ii) $dg = \omega.n + \omega_1.\Delta$. When we substitute the first in the second equation, we get $d(\alpha.n) - \omega.n = -d(\alpha_1.\Delta) + \omega_1.\Delta$ Hence indeed $n \in P_\Sigma(\mathcal{A}) \Leftrightarrow w(n) = 0$, which proves the first statement. Further one has : $(0,g) \in \mathcal{A} \Leftrightarrow \exists\ \alpha_1, \omega_1$ such that i) $g = \alpha_1.\Delta$ and ii) $dg = \omega_1.\Delta$, i.e. $g \in \int I/(f)$. For the last statement, remark that by the exact sequence (2) one has that the equation $w(n) = -d(\alpha_1.\Delta) + \omega_1.\Delta$ determines the element $\alpha_1.\Delta$ modulo $\int I$. One then checks that the element $\varphi(n) = \alpha.n + \alpha_1.\Delta$ is well- defined modulo $\int I$. ⊠

To get from the space $\mathcal{A}$ of embedded admissible pairs to the space $T^1(\Sigma,X)$ of admissible first order deformations, we have to divide out by the infinitesimal automorphisms, which are generated by the vector fields $\vartheta \in \Theta$ on the ambient space:

$$T^1(\Sigma,X) = \mathcal{A} / \{ (\vartheta(\Delta),\vartheta(f)) | \vartheta \in \Theta \} \quad (10)$$

In (3.5) $\mathcal{A}$ appears as an extension of $P_\Sigma(\mathcal{A})$ by $\int I/(f)$ and this suggests that we first divide out the automorphisms that act on $P_\Sigma(\mathcal{A})$ and then let act $\Theta(\log\Sigma) := \{\vartheta \in \Theta \mid \vartheta(I) \subset I\}$ on $\int I/(f)$.

First we have to check that indeed our maps α, ω and w descend from the space N_Σ to T^1_Σ.

Lemma (3.6) :

The α and the ω - map (and hence the w - map) descend to maps

$$\alpha : T^1_\Sigma \longrightarrow \mathcal{O}_\Sigma$$
$$\omega : T^1_\Sigma \longrightarrow \Omega^1_\Sigma$$

(which we continue to denote by the same symbols).

proof : By lemma (3.2) we have $\alpha . d\Delta + \Gamma . \Delta = 0$ for some Γ. Contracting this equation with $\vartheta \in \Theta$ then gives $\alpha . \vartheta(\Delta) \in I$. This means that the α - map descends. Let $\mathcal{L}_\vartheta$ denote the Lie-derivative with respect to ϑ. From $df = \omega . \Delta$ we then get

$$d(\vartheta(f)) = \mathcal{L}_\vartheta(df) = \mathcal{L}_\vartheta(\omega) . \Delta + \omega . \vartheta(\Delta) \quad (11)$$

and as $\vartheta(f) \in I$ we indeed see $\omega . \vartheta(\Delta) = 0$ in Ω^1_Σ. ⊠

Definition / Corollary (3.7) :

Define $T^1_\Sigma(X) := \mathrm{Im}(T^1(\Sigma,X) \longrightarrow T^1_\Sigma)$. Then:

i) There are exact sequences

$$0 \longrightarrow T^1_\Sigma(X) \longrightarrow T^1_\Sigma \xrightarrow{\;w\;} \Omega^1_\Sigma$$

$$0 \longrightarrow \int I/(f, J_\Sigma(f)) \longrightarrow T^1(\Sigma,X) \longrightarrow T^1_\Sigma(X) \longrightarrow 0$$

Here $J_\Sigma(f) := \{\vartheta(f) \mid \vartheta(I) \subset I\}$.

ii) There is a natural map $\varphi : T^1_\Sigma(X) \longrightarrow \mathcal{O}/(\int I + J(f))$ which associates to an $[n] \in T^1_\Sigma(X)$ a unique coset $\varphi([n])$ such that

$$[(n,g)] \in T^1(\Sigma,X) \Leftrightarrow g \in \varphi([n]) + \int I + J(f).$$

proof : This follows immediately from (3.5) by dividing out the vector fields. Note that from (11) it follows that $\vartheta(I) \subset I \Rightarrow \vartheta(\int I) \subset \int I$. ⊠

Remark (3.8) :

R. Pellikaan ([Pe 3], pp. 19-32) studied the slightly different problem of admissible deformations of a *map* f . The space $J_\Sigma(f)$ he calls the *extended I - tangent space* to f and the number $c_{I,e}(f) = \dim_{\mathbb{C}} (\int I/J_\Sigma(f))$ he calls the *extended I - codimension.* It is clear that in our situation the space $\int I/(f, J_\Sigma(f))$ is the tangent space to the functor of admissible deformations for which Σ is kept *fixed.*

We know from § 2. that under reasonable assumptions that one has that $\mathrm{Def}(\Sigma, X) \hookrightarrow \mathrm{Def}(X)$ and hence $T^1(\Sigma,X) \hookrightarrow T^1_X$.

Proposition (3.9) :

Assume that $T^1(\Sigma,X) \hookrightarrow T^1_X$. Then one has:

i) $\int I/(f,J_\Sigma(f)) \xrightarrow{=} \int I/\int I \cap (f,J(f)) \ \left(\approx (\int I + J(f))/(f,J(f))\right)$

ii) $\varphi : T^1_\Sigma(X) \hookrightarrow \mathcal{O}/(\int I + J(f))$

iii) $T^1(\Sigma,X) \approx (\varphi(T^1_\Sigma(X)) + \int I + J(f))/(f,J(f)) \subset T^1_X$.

proof : i) $\int I/(f,J_\Sigma(f))$ is a subspace of $T^1(\Sigma,X)$. If this is to inject in T^1_X , then the map in i) is injective. As it is clearly surjective, it is an isomorphism. (c.f. [Pe 3], prop. 5.3). The injectivity in ii) expresses just the fact that if $[(n,g)] \in T^1(\Sigma,X)$, and $T^1(\Sigma,X) \hookrightarrow T^1_X$, then the deformation $[n]$ of Σ is essentially determined by g. Statement iii) then follows from i) and ii). ⊠

The following proposition gives the dependence of the w - map on our function $f \in \int I$.

Proposition (3.10) :

i) There is a $\mathbb{C}$- bilinear map

$$W : \int I/I^2 \times T^1_\Sigma \longrightarrow \Omega^1_\Sigma$$
$$(f \ , \ n) \longmapsto w_f(n)$$

where $w_f(n) = d(\alpha.n) - \omega.n$, $f = \alpha.\Delta$ and $df = \omega.\Delta$.

ii) If $f \in I^2$ (so $f = (h.\Delta).\Delta$ for some symmetric matrix h) then we have $P_\Sigma(\mathcal{A}) = N_\Sigma$ and the induced map $\varphi : N_\Sigma \longrightarrow \mathcal{O}/\int I$

is given by $\quad n \longmapsto 2.(h.\Delta).n$

proof : The above map is clearly linear in f. We show that if $f \in I^2$, then w_f is the zero map. But if $f = h.\Delta.\Delta$ for some matrix h, then we can take $\alpha = h.\Delta$ and $\omega = d(h).\Delta + 2.h.d\Delta$, so $w_f(n) = 0$ in Ω^1_Σ. ⊠

Above we considered the space $T^1(\Sigma,X)$ for a general Σ. In the situation we are most interested in there is an important simplification.

Lemma (3.11) :

Assume that Σ is *reduced*. Then the α - map

$$\alpha : T^1_\Sigma \longrightarrow \mathcal{O}_\Sigma$$

is the zero map.

proof : If Σ is reduced, then T^1_Σ is a torsion $\mathcal{O}_\Sigma$ - module. As $\mathcal{O}_\Sigma$ is torsion free, α has to be the zero map. ⊠

The α - map being the zero map has the effect of making all maps we encountered not only ($\mathbb{C}$-) linear, but even *module homomorphisms.*

Proposition (3.12) :

If the α - map is the zero map one has:

i) $(n,g) \in \mathcal{A} \Rightarrow g \in I$.

ii) $w : N_\Sigma \longrightarrow \Omega^1_\Sigma$ is $\mathcal{O}_\Sigma$ - linear.

iii) $\varphi : P_\Sigma(\mathcal{A}) \longrightarrow I/\int I$ is $\mathcal{O}_\Sigma$ - linear.

iv) $\mathcal{A}$ and $T^1(\Sigma,X)$ are $\mathcal{O}_X$ - modules.

v) $P_\Sigma(\mathcal{A})$ and $T^1_\Sigma(X)$ are $\mathcal{O}_\Sigma$ - modules.

We omit the easy and straight forward proof.

Remark (3.13) :

In general there is an exact sequence of the form

$$0 \longrightarrow N_{\Sigma,X} \longrightarrow N_{\Sigma} \xrightarrow{\alpha} \mathcal{O}_{\Sigma}$$

where $N_{\Sigma,X} = \mathrm{Hom}_X(I \otimes \mathcal{O}_X, \mathcal{O}_{\Sigma})$ is the normal bundle of Σ inside X. So the equality of these two normal bundles is in fact equivalent to the α - map being the zero map. In § 4. we will encounter another interpretation of the condition that α is the zero map.

Corollary (3.14) :

If Σ is reduced, then the map $w: T^1_{\Sigma} \longrightarrow \Omega^1_{\Sigma}$ in fact lands in the torsion sub-sheaf $\mathrm{Tors}(\Omega^1_{\Sigma})$, i.e. $w : T^1_{\Sigma} \longrightarrow \mathrm{Tors}(\Omega^1_{\Sigma})$.

proof : This is clear, because w is $\mathcal{O}_{\Sigma}$ - linear and T^1_{Σ} is torsion. ⊠

We summarize the above discussion in a theorem.

Theorem (3.15) :

Let Σ be defined by an ideal I and let $f \in \int I$. Assume that:

i) Σ is reduced and Cohen - Macaulay.

ii) $\dim(\Sigma) \geq 1$.

iii) $\dim(I/(f,J(f)) < \infty$.

Then one has:

$$T^1(\Sigma,X) \approx P_X(\mathcal{A})/(f,J(f)) \subset I/(f,J(f)) \subset T^1_X .$$

where $P_X(\mathcal{A})$ is the *ideal* $\left(\varphi(T^1_{\Sigma}(X)) + \int I + J(f)\right)$

and $T^1_{\Sigma}(X) = \mathrm{Ker}(\, w : T^1_{\Sigma} \longrightarrow \mathrm{Tors}(\Omega^1_{\Sigma}))$.

proof : Use (2.4), (3.9),(3.11),(3.12) and (3.14). ⊠

We conclude this part with some simple examples.

Examples (3.16) :

1) $f = xyz \in \mathbb{C}\{x,y,z\}$, $I = (y.z,\ z.x,\ x.y)$. Because $I/J(f) = 0$ we have $T^1(\Sigma,X) = 0$. In this example one has $w : T^1_{\Sigma} \xrightarrow{\approx} \mathrm{Tors}(\Omega^1_{\Sigma})$ and $T^1_{\Sigma}(X) = 0$.

2) $f = xy^2 \in \mathbb{C}\{x,y\}$, $I = (y)$.

Then $P_X(\mathcal{A}) = (xy, y^2) = J(f)$, hence $T^1(\Sigma, X) = 0$.

3) $f = x^2y^2 + y^4$, $I = (y)$.

Then $P_X(\mathcal{A}) = (x^2y, y^2)$ and $J(f) = (2x^2y + 4y^3, xy^2)$

Hence $T^1(\Sigma,X)$ is 2 - dimensional.

4) $f = (yz)^2 + (zx)^2 + (xy)^2$; $I = (yz,zx,xy)$.

Because $f \in I^2$, $w : N_\Sigma \longrightarrow \Omega^1_\Sigma$ is the zero map. Hence $P_\Sigma(\mathcal{A}) = N_\Sigma$ and is generated by the following vectors:

$(y,0,0)$, $(z,0,0)$, $(0,x,0)$, $(0,z,0)$, $(0,0,x)$, $(0,0,y)$.

A calculation shows that:

$P_X(\mathcal{A}) = (y^2z, yz^2, z^2x, zx^2, x^2y, xy^2, xyz)$ and

$(f, J(f)) = (xy^2+xz^2, x^2z+y^2z, x^2y+z^2y)$.

Hence $\dim T^1(\Sigma,X) = 7$, with as basis:

$\{3xyz, 2(y^2z+yz^2), 2(x^2z+xz^2), 2(x^2y+xy^2), 2x^2yz, 2xy^2z, 2xyz^2\}$.

Conjecture (3.17) :

Let X be a germ of a weakly normal surface singularity in $\mathbb{C}^3$ with singular locus Σ. Then:

$$
\begin{aligned}
T^1(\Sigma,X) = 0 \Leftrightarrow X &\approx A_\infty &&: f = y^2 + z^2 \quad \text{or}\\
&\approx D_\infty &&: f = x.y^2 + z^2 \quad \text{or}\\
&\approx T_{\infty,\infty,\infty} &&: f = xyz
\end{aligned}
$$

Remark (3.18) :

We will see in § 4 the reason why one in general can *not* expect that for every $\Sigma \hookrightarrow X$ there is an admissible deformation such that on the general fibre the space X_η has only singularities as in (3.17). Such a deformation we call a *disentanglement* (see (5.6). In fact, if $\mathcal{E} \subset \mathbb{P}^2$ is a curve with ordinary nodes which is birational to a non-hyperelliptic curve of sufficiently high degree, then $X = \mathrm{Cone}(\mathcal{E})$, $\Sigma = \mathrm{Sing}(X)_{red}$ is an example of a pair $\Sigma \hookrightarrow X$ which has essentially only 'equisingular' admissible deformations (c.f. [Mu]). In particular, it can not have a disentanglement.

B. Obstruction Theory and Semi - Universal Deformation.

In this part we consider the problem of extending a given admissible deformation over a space to a slightly bigger space. We will show that in our situation the theory is completely analogous to the ordinary deformation theory. We also outline the steps that lead to the construction of the base space of the semi - universal admissible deformation. As an application we prove a theorem stating that this base space only depends, up to a smooth factor, on the class of $f \in \int I$ modulo I^2.

Consider a 'small surjection' of rings $S' \longrightarrow S$, i.e. suppose we are given an exact sequence of the form:

$$0 \longrightarrow V \longrightarrow S' \longrightarrow S \longrightarrow 0 \qquad (12)$$

where V is an ideal in S' with the property $V.m_{S'} = 0$. In this situation V becomes an S - module, in fact a module over $k = S/m_S$. We study the map $\rho : Def(\Sigma,X)(S') \longrightarrow Def(\Sigma,X)(S)$. The questions that arize are:

* What is the image of ρ, i.e. which deformations over S can be extended to deformations over S' ?
* What are the fibres of ρ, i.e. in how many different ways can one extend a given deformation over S to one over S' ?

Given a deformation $\Sigma_S \hookrightarrow X_S \in Def(\Sigma,X)(S)$, we split up the above problem into three steps.

1. Try to lift Σ_S to $\Sigma_{S'}$ over S' . This is an ordinary deformation problem for Σ.
2. Given a lift $\Sigma_{S'}$ of Σ_S , try to find $X_{S'}$ as to make an admissible deformation with $\Sigma_{S'}$.
3. Vary the choice of $\Sigma_{S'}$ in 2. to get from a given $\Sigma_S \hookrightarrow X_S \in Def(\Sigma,X)(S)$ to an element $\Sigma_{S'} \hookrightarrow X_{S'} \in Def(\Sigma,X)(S')$.

About step 1. we quote the following theorem.

Theorem (3.19) :

Given a deformation $\Sigma_S \in \mathrm{Def}(\Sigma)(S)$ and a small surjection as in (12), then there is an element

$$\mathrm{Ob}(\Sigma_S) \in T^2_\Sigma \otimes V$$

(where $T^2_\Sigma := \mathrm{Hom}(\mathcal{R}/\mathcal{R}_0, \mathcal{O}_\Sigma)/\mathrm{Hom}(\mathcal{F}, \mathcal{O}_\Sigma)$, where $\mathcal{R}$, $\mathcal{F}$ as in (6) and $\mathcal{R}_0$ is the module of 'Koszul relations') with the following properties:

i) Σ_S extends to an $\Sigma_{S'}$ $\Leftrightarrow$ $\mathrm{Ob}(\Sigma_S) = 0$.

ii) If $\mathrm{Ob}(\Sigma_S) = 0$, then the possible choices for $\Sigma_{S'}$ form a principal homogeneous space over $T^1_\Sigma \otimes V$.

proof : Well - known, see [Schl 2], pp.149-150. ⊠

(We use $\bigotimes$ for tensor products over the ring and $\otimes$ for tensor products over the ground field k.)

Given an admissible deformation $\Sigma_S \hookrightarrow X_S \in \mathrm{Def}(\Sigma, X)(S)$ we thus get a *first obstruction* $\mathrm{Ob}(\Sigma_S) \in T^2_\Sigma \otimes V$. If this obstruction vanishes we choose an $\Sigma_{S'}$ and go on with step 2. for which we have the following.

Proposition (3.20) :

Given an admissible deformation $\xi_S = \left(\Sigma_S \hookrightarrow X_S\right) \in \mathrm{Def}(\Sigma, X)(S)$ and a lift $\Sigma_{S'}$ of Σ_S, then there exists an element

$$\mathrm{Ob}(\xi_S, \Sigma_{S'}) \in \Omega^1_\Sigma \otimes V$$

with the following properties:

i) There is an $X_{S'}$ such that $\left(\Sigma_{S'} \hookrightarrow X_{S'}\right) \in \mathrm{Def}(\Sigma, X)(S')$ $\Leftrightarrow$ $\mathrm{Ob}(\xi_S, \Sigma_{S'}) = 0$.

ii) If $\mathrm{Ob}(\xi_S, \Sigma_{S'}) = 0$, then the possible choices for $X_{S'}$ form a principal homogeneous space over $\int I/(f, J_\Sigma(f)) \otimes V$.

proof : Let $P_S = \mathcal{O}_{\mathbb{C}^{n+1} \times S}$ the local ring of the ambient space over S and let $P_{S'}$ be defined analogous. Then an element ξ_S is represented by α_S and Γ_S such that (see (3.2)) :

$$\alpha_S . \Delta_S = f_S$$
$$\alpha_S . d\Delta_S + \Gamma_S . \Delta_S = 0$$

Here Σ_S is defined by the ideal $I_S = (\Delta_S)$ and X_S by f_S.
Note that there is an exact sequence of Kähler differentials:

$$0 \longrightarrow V \otimes \Omega^1_{P_{S'}/S'} \longrightarrow \Omega^1_{P_{S'}/S'} \longrightarrow \Omega^1_{P_S/S} \longrightarrow 0$$

and an isomorphism $V \otimes \Omega^1_{P_{S'}/S'} \xrightarrow{\approx} V \otimes \Omega_P$, $\Omega_P = \Omega^1_{\mathbb{C}^{n+1}}$.

Now given is a lift $\Sigma_{S'}$ of Σ_S , i.e. we have $\Delta_{S'}$. Take *any* lift of α_S to $\alpha_{S'}$ and Γ_S to $\Gamma_{S'}$ and consider the element

$$w_{S'} = \alpha_{S'} . d\Delta_{S'} + \Gamma_{S'} . \Delta_{S'} \in \Omega^1_{P_{S'}/S'}$$

Because over S $w_{S'}$ is the zero - form, we see that via the above isomorphism we can consider $w_{S'}$ as an element of $V \otimes \Omega^1_P$.

Claim : The class of $w_{S'}$ in $\Omega^1_\Sigma \otimes V$ is independent of the choice of $\alpha_{S'}$ and $\Gamma_{S'}$. In fact, the differences of two choices of $\alpha_{S'}$ and $\Gamma_{S'}$ are of the form $v_1 \otimes \beta_{S'}$ and $v_2 \otimes \gamma_{S'}$, $v_i \in V$. Hence the difference of the $w_{S'}$ is of the form $v_1 \otimes \beta_{S'}.d\Delta_{S'} + v_2 \otimes \gamma_{S'}.\Delta_{S'} = v_1 \otimes \beta.d\Delta + v_2 \otimes \gamma\Delta$, because $V.m_{S'} = 0$. Consequentely, the class of $w_{S'}$ in $\Omega^1_\Sigma \otimes V$ is only dependent on $\xi_S = (\Sigma_S \hookrightarrow X_S)$ and $\Sigma_{S'}$. We put:

$$Ob(\xi_S, \Sigma_{S'}) = [w_{S'}] \in \Omega^1_\Sigma \otimes V .$$

From the exact sequence (2)

$$0 \longrightarrow I/\textstyle\int I \xrightarrow{\ d\ } \Omega^1_P \otimes \mathcal{O}_\Sigma \longrightarrow \Omega^1_\Sigma \longrightarrow 0$$

we see that extension to S' is possible if and only $[w_{S'}] = 0$ and that then the choice $f_{S'} = \alpha_{S'} . \Delta_{S'}$ is determined modulo $\int I \otimes V$. Dividing out the isomorphisms for $\Sigma_{S'} \hookrightarrow X_{S'}$ then leads to a principal homogeneous space over $\int I/(f, J_\Sigma(f)) \otimes V$, as in (3.7). ⊠

Finally, the result about step 3. is the following.

Theorem (3.21) :

Let $\xi_S = \left(\Sigma_S \hookrightarrow X_S \right) \in \mathrm{Def}(\Sigma,X)(S)$ be an admissible deformation over S. Then one has the following:

i) There is a *first obstruction* $\mathrm{Ob}(\Sigma_S) \in T^2_\Sigma \otimes V$.

ii) If $\mathrm{Ob}(\Sigma_S) = 0$, then there is a *second obstruction* $\mathrm{Ob}(\xi_S) \in T^2(\Sigma,X) \otimes V$.

iii) ξ_S lifts to an element $\xi_{S'} = \left(\Sigma_{S'} \hookrightarrow X_{S'}\right) \in \mathrm{Def}(\Sigma,X)(S') \Leftrightarrow$ $\mathrm{Ob}(\Sigma_S) = 0$ and $\mathrm{Ob}(\xi_S) = 0$.

iv) The possible choices for $\xi_{S'}$ in iii) form a principal homogeneous space over $T^1(\Sigma,X) \otimes V$.

Here $T^2(\Sigma,X) := \mathrm{Coker}(w : T^1_\Sigma \longrightarrow \Omega^1_\Sigma)$ is the *obstruction space* of our problem.

proof : This readily follows from (3.19) and (3.20). We only have to study the behaviour of the class $\mathrm{Ob}(\xi_S, \Sigma_{S'})$ when we vary the lift $\Sigma_{S'}$ of Σ_S. Two different choices of $\Delta_{S'}$ differ by an element $n \in N_\Sigma \otimes V$, so $\mathrm{Ob}(\xi_S, \Sigma_{S'})$ descends to a class $\mathrm{Ob}(\xi_S) \in T^2(\Sigma,X) \otimes V$ as defined above. (We omit some further details.) ⊠

Remark (3.22) :

If the α - map of (3.3) is the zero - map (c.f. (3.11)), then $T^2(\Sigma,X)$ is not only a vector space, but also an $\mathcal{O}_\Sigma$ - module. In fact, in that case $w : T^1_\Sigma \longrightarrow \Omega^1_\Sigma$ is $\mathcal{O}_\Sigma$ - linear and $T^2(\Sigma,X) = \mathrm{Coker}(w)$.

When we assume that Σ is reduced then T^1_Σ is a torsion $\mathcal{O}_\Sigma$ - module. In that case it can be seen that the obstruction $\mathrm{Ob}(\xi_S)$ in fact lands in the $\mathcal{O}_\Sigma$ - module $\mathrm{Tors}(\Omega^1_\Sigma)/w(T^1_\Sigma)$. In the case that Σ has an isolated singular point, this is a finite dimensional vector space.

We now apply Schlessingers method (see [Schl1], pp. 213-215) to construct the *hull* of the functor $\mathrm{Def}(\Sigma, X)$. For reasons of simplicity we will assume that Σ is a reduced space with an isolated singular point (see (3.22)) and that $T^2_\Sigma = 0$. (For example, Σ could be the germ of a space curve in $\mathbb{C}^3$.)

Procedure (3.23) :

* Take a basis $t_1, t_2, \ldots, t_\tau$ of the dual space of $T^1(\Sigma, X)$, and a basis $v_1, v_2, \ldots, v_\sigma$ for the dual space of $\mathrm{Tors}(\Omega^1_\Sigma)/w(T^1_\Sigma)$.
* Consider the ring $\mathfrak{U} := \mathbb{C}[\![t_1, t_2, \ldots, t_\tau]\!]$. We are going to define a sequence of ideals $J_2 \supset J_3 \supset \ldots \supset J_q \supset \ldots$ in $\mathfrak{U}$, together with elements $\xi_q \in \mathrm{Def}(\Sigma,X)(\mathfrak{U}_q)$, where $\mathfrak{U}_q = \mathfrak{U}/J_q$.
* The ideals J_q and the deformations ξ_q are defined inductively. One puts $J_2 = m^2$ and constructs the universal first order family ξ_2. In term of equations ξ_2 is given by $\alpha_1^{(i)}$, $\Gamma_1^{(i)}$ and $n_1^{(i)}$ such that
  $\left(f + \sum t_i . g^{(i)}\right) = \left(\alpha + \sum t_i . \alpha_1^{(i)}\right).\left(\Delta + \sum t_i . n_1^{(i)}\right)$ and
  $0 = \left(\alpha + \sum t_i . \alpha_1^{(i)}\right).d\left(\Delta + \sum t_i . n_1^{(i)}\right) + \left(\Gamma + \sum t_i . \Gamma_1^{(i)}\right).\left(\Delta + \sum t_i . n_1^{(i)}\right)$
  where both equations are mod m^2.

  Now assume we have constructed $\xi_q \in \mathrm{Def}(\Sigma,X)(\mathfrak{U}_q)$. We look for an ideal J_{q+1}, $m.J_q \subset J_{q+1} \subset J_q$ which is *minimal* with respect to the property that ξ_q extends to an $\xi_{q+1} \in \mathrm{Def}(\Sigma,X)(\mathfrak{U}_{q+1})$.
* A way to obtain such a J_{q+1} is as follows. Consider the small surjection

  $$0 \longrightarrow J_q/m.J_q \longrightarrow \mathfrak{U}/m.J_q \longrightarrow \mathfrak{U}_q \longrightarrow 0$$

  We get an obstruction $\mathrm{Ob}(\xi_q) \in \mathrm{Tors}(\Omega^1_\Sigma)/w(T^1_\Sigma) \otimes \left(J_q/m.J_q\right)$. Applying the elements $v_1, v_2, \ldots, v_\sigma$ to $\mathrm{Ob}(\xi_q)$ we get elements $V_1, V_2, \ldots, V_\sigma$ in $J_q/m.J_q$. It is now clear that one can take $J_{q+1} = (V_1, V_2, \ldots, V_\sigma, m.J_q)$. Now one *chooses any* ξ_{q+1} *lifting the element* ξ_q and then considers $\mathrm{Ob}(\xi_{q+1})$ etc.

One easily sees by induction from the above construction that the ideal J_q is generated by σ elements V_1 , V_2 , . . . , V_σ and m^q . We call the elements V_i mod m^q , i = 1, 2, . . . , σ, *the equations of the base space of the semi - universal admissible deformation to order* $q-1$ or simply *the equations of the base space.* It should be stressed that these equations do not only depend on the chosen bases t_i and v_j , but also on the chosen lifts ξ_q . In practice this is of crucial importance: good choices can simplify the equations a lot, whereas bad choices make the computations into a nightmare.

Example (3.24) :

This example is a continuation of example (3.16) 4). As the map $w : N_\Sigma \longrightarrow \Omega^1_\Sigma$ is the zero map, we have that the obstruction space is $\mathrm{Tors}(\Omega^1_\Sigma)$. One easily computes that the dimension of this space is three and has as a basis:

$$w_1 = x.d(y+z);\ w_2 = y.d(z+x);\ w_3 = z.d(x+y)\ .$$

The semi-universal deformation to first order is described by the following data:

a) the deformed curve: $\Delta_1 = \Delta + \sum_{i=1}^{3} t_i.n_i$, where

$\Delta = (yz,zx,xy);\ n_1 = (y+z,0,0);\ n_2 = (0,z+x,0);\ n_3 = (0,0,x+y)$.

b) the deformed α's: $\alpha_1 = h_1.\Delta_1 + t_0.(x,y,z)$, where

$$h_1 = \begin{pmatrix} 1 & t_4 & t_5 \\ t_4 & 1 & t_6 \\ t_5 & t_6 & 1 \end{pmatrix}$$

c) the deformed f is $f_1 = \alpha_1.\Delta_1$ (mod m^2).

d) the deformed Γ's : $\Gamma_1 = -\ h_1.d\Delta_1 - 2t_0.(dx,dy,dz)$.

The curve is not obstructed, and a lift to second order is given by:

$$\Delta_2 = \Delta_1 + \sum_{i\neq j}^{3} t_i.t_j(1,1,1)$$

The obstruction element $\alpha_1 d\Delta_2 + \Gamma_1.\Delta_2$ in $\mathrm{Tors}(\Omega^1_\Sigma)\otimes(m^2/m^3)$ is

$$t_0.(x,y,z).d(\textstyle\sum_{i=1}^{3} t_i.n_i) - 2t_0.(dx,dy,dz).(\sum_{i=1}^{3} t_i.n_i)$$

Because in Ω^1_Σ we have the relation $(y+z)dx+xd(y+z)=d(xy+xz)=0$, we can rewrite this expression as:

$$3t_0t_1 \cdot w_1 + 3t_0t_2 \cdot w_2 + 3t_0t_3 \cdot w_3 \in \mathrm{Tors}(\Omega^1_\Sigma) \otimes (m^2/m^3) \ .$$

Hence the equations for the base to second order are given by:

$$t_0t_1 = 0 \ ; \qquad t_0t_2 = 0 \ ; \qquad t_0t_3 = 0 \ .$$

A lift of f to second order is given by $f_2 = (h_1 \cdot \Delta_2).\Delta_2 + t_0\,(x,y,z).\Delta_2$ and one can check that this family, as it stands, defines an admissible family to every order.

Suppose that we are given admissible diagrams $\Sigma \hookrightarrow X^{(1)}$ and $\Sigma \hookrightarrow X^{(2)}$, where $X^{(i)}$ is described by $f^{(i)} \in \int I$. When $f^{(1)} - f^{(2)} \in I^2$, then (3.10) implies that the first order obstruction map for $X^{(1)}$ and $X^{(2)}$ are 'the same'. In fact, the following propositions show that when $f^{(1)} - f^{(2)} \in I^2$ something much stronger is true: $\mathrm{Def}(\Sigma,X^{(1)})$ and $\mathrm{Def}(\Sigma,X^{(2)})$ are the same up to a smooth factor.

Proposition (3.25) :

Let $\xi_S^{(i)} = (\Sigma_S \hookrightarrow X_S^{(i)}) \in \mathrm{Def}(\Sigma,X^{(i)})$, i = 1,2 and let be given a small surjection as in (12).

Let $X_S^{(i)}$ be defined by $f_S^{(i)} \in \int I_S$. If $f_S^{(1)} - f_S^{(2)} \in I_S^2$ then we have:

i) $T^2(\Sigma,X^{(1)}) = T^2(\Sigma,X^{(2)})$.

ii) $\mathrm{Ob}(\xi_S^{(1)}) = \mathrm{Ob}(\xi_S^{(2)})$.

iii) Let $S' \longrightarrow S$ any surjection. Then $\xi_S^{(1)}$ can be lifted to S' if and only if $\xi_S^{(2)}$ can be lifted to S'. If this is the case, we can do this in such a way that $f_{S'}^{(1)} - f_{S'}^{(2)} \in I_{S'}^2$.

proof : We have $T^2(\Sigma,X^{(i)}) = \mathrm{Coker}(w^{(i)} : T^1_\Sigma \longrightarrow \Omega^1_\Sigma)$, and by (3.10) we have $w^{(1)} = w^{(2)}$, hence we get i). For ii) we assume that Σ_S is lifted to $\Sigma_{S'}$. As $f_S^{(1)} - f_S^{(2)} = h_S \cdot \Delta_S \cdot \Delta_S$ for some matrix h_S , we can take $\alpha_S^{(1)} - \alpha_S^{(2)} = h_S \cdot \Delta_S$ and $\Gamma_S^{(1)} - \Gamma_S^{(2)} = h_S \cdot d\Delta_S$. Now lift h_S to a matrix $h_{S'}$ and $\alpha_S^{(1)}$, $\Gamma_S^{(1)}$ over S'. Define then the lifts for $\alpha_S^{(2)}$, $\Gamma_S^{(2)}$ by requiring the above relations to hold over S'. Then one has $w_{S'}^{(1)} = w_{S'}^{(2)}$, so $\mathrm{Ob}(\xi_S^{(1)}) = \mathrm{Ob}(\xi_S^{(2)})$. Statement iii) can be deduced from ii) by factoring the surjection in a sequence of small surjections. The indicated choices above lead to $f_{S'}^{(1)} - f_{S'}^{(2)} \in I_{S'}^2$. ⊠

Definition (3.26) :

Two deformations $\xi_S^{(i)}=(\Sigma_S,X_S^{(i)}) \in \mathrm{Def}(\Sigma,X)$ $(i=1,2)$ are called *I^2-equivalent* if there are equations $f_S^{(i)}$ for $X_S^{(i)}$ such that $f_S^{(1)} - f_S^{(2)} \in I_S^2$, where I_S is the ideal of Σ_S. We denote this equivalence relation by $\sim$.
The functor

$$\begin{array}{ccc} \mathbf{C} & \longrightarrow & \mathbf{Set} \\ S & \longmapsto & \mathrm{Def}(\Sigma,X)/\sim \end{array}$$

is denoted by $M(\Sigma,X)$ and is called the functor of *admissible deformations modulo I^2* .

Proposition (3.27) :

Let $\Sigma \hookrightarrow X$ be an admissible diagram. Then one has:

i) The natural transformation $\mathrm{Def}(\Sigma,X) \longrightarrow M(\Sigma,X)$ is smooth.

ii) $M(\Sigma,X)$ is a semi-homogeneous functor.

iii) If $X^{(1)}$ is I^2- equivalent to $X^{(2)}$, then there is a natural equivalence of functors $M(\Sigma,X^{(1)}) \approx M(\Sigma,X^{(2)})$.

iv) The space $M^1(\Sigma,X) := M(\Sigma,X)(k[\varepsilon]/(\varepsilon^2))$ fits in an exact sequence:

$$0 \longrightarrow \int I/\big((f,J_\Sigma(f)) + I^2\big) \longrightarrow M^1(\Sigma,X) \longrightarrow T^1_\Sigma(X) \longrightarrow 0$$

Sketch of proof : Statement i) follows from (3.25). Statement ii) can be proved by showing that $\sim$ is an 'admissible' equivalence relation in the sense of [Bu], p23. (The word *admissible* here should not confuse the reader.) For this one uses i). Statement iii) is essentially trivial: the equivalence of functors is defined in the obvious way on representatives. The fact that $\sim$ is really an equivalence relation then shows that it is well-defined. Statement iv) is proved along the lines of (3.7)i). Note that by (3.10)the space T^1_Σ (X) depends indeed only on the class of f modulo I^2 . It is easy to see that the first space in the exact sequence iv) depends only on $[f] \in \int I/I^2$. ⊠

Corollary (3.28) :

If $\int I/(I^2 + f^{(1)}) = \int I/(I^2 + f^{(2)})$, then the base space of $\Sigma \hookrightarrow X^{(1)}$ and $\Sigma \hookrightarrow X^{(2)}$ are the same up to a smooth factor.

C. *Obstructions and the Hessian.*

From now on we consider an admissible diagram $\Sigma \hookrightarrow X$ with Σ *reduced.* In B. we have seen that the obstruction space for the functor $\mathrm{Def}(\Sigma,X)$ is, apart from T^2_Σ , the $\mathcal{O}_\Sigma$ - module $\mathrm{Tors}(\Omega^1_\Sigma)/w(T^1_\Sigma)$. This space however seems to be too big in general. For example, when Σ is a reduced complete intersection, then $\int I = I^2$ and hence we know that all obstructions vanish. But $\mathrm{Tors}(\Omega^1_\Sigma) \neq 0$ (unless Σ is smooth) and $w(T^1_\Sigma) = 0$, so the obstruction space is never zero. In fact, it is a long standing conjecture that for a singular curve germ one always has $\mathrm{Tors}(\Omega^1_\Sigma) \neq 0$. (Berger's conjecture, see [Be].) It turns out that in the case that $T^2_\Sigma = 0$ there is a subspace of $\mathrm{Tors}(\Omega^1_\Sigma)$ which receives all the obstructions. Let us first describe this subspace, which we call N^*/I. Recall the exact sequence (2) of $\mathcal{O}_\Sigma$ - modules:

$$0 \longrightarrow I/\int I \xrightarrow{\ d\ } \Omega^1 \otimes \mathcal{O}_\Sigma \longrightarrow \Omega^1_\Sigma \longrightarrow 0$$

Now take the double dual of the map d. As $\mathrm{Hom}_\Sigma(I/\int I, \mathcal{O}_\Sigma) = N_\Sigma$ and $\Omega^1 \otimes \mathcal{O}_\Sigma$ is $\mathcal{O}_\Sigma$ - free, we get:

$$0 \longrightarrow N_\Sigma^* \xrightarrow{\ d^{**}\ } (\Omega^1 \otimes \mathcal{O}_\Sigma)^{**} \approx \Omega^1 \otimes \mathcal{O}_\Sigma$$

where $N_\Sigma^* = \mathrm{Hom}_\Sigma(N_\Sigma, \mathcal{O}_\Sigma)$ is the *dual* of the normal bundle. Further, there is the double duality inclusion $I/\int I \hookrightarrow N_\Sigma^*$ and hence we get an inclusion:

$$N^*/I := N^*_\Sigma/(I/\int I) \hookrightarrow \mathrm{Tors}(\Omega^1_\Sigma) \subset \Omega^1_\Sigma \qquad (13)$$

(N^*/I is a torsion $\mathcal{O}_\Sigma$ - module, so it lands in $\mathrm{Tors}(\Omega^1_\Sigma)$.)
Note that if Σ is a complete intersection, then I/I^2 is a free $\mathcal{O}_\Sigma$ - module, and hence $N^*/I = 0$.

Conjecture (3.29) :

$N^*/I = 0$ if and only if Σ is a complete intersection.

We have to admit however that we do not have overwhelming evidence for the truth of this conjecture. In practice it is hard to compute the space N^*/I. If Σ is Cohen-Macaulay of codimension 2, one can prove that $N^*/I \approx \mathcal{E}xt^2_\Sigma(\omega_\Sigma, \mathcal{O}_\Sigma)$, but this does not seem to be of great help to settle the conjecture even for space curves. Note that conjecture (3.29) implies Berger's conjecture, because for a complete intersection curve singularity Berger's conjecture is known to be true.

Lemma (3.30) :

Via the canonical injection $I/\int I \hookrightarrow N_\Sigma^*$ the element $\beta.\Delta \in I$ is send to the homomorphism β: $N_\Sigma \longrightarrow \mathcal{O}_\Sigma$; $n \longmapsto \beta.n$.

proof : Disentangle the double duality definition. ⊠

From the above we see that for an $f \in I$ one has (c.f. (3.11):

$$f \in \int I \iff \text{the } \alpha \text{ - map of } f \text{ is the zero map.}$$

As the above statement can be made to work over any base S, we get an alternative way to express the condition that $\Sigma_S \hookrightarrow X_S$ is an *admissible* deformation. First we define the *relative primitive ideal* $\int I_S$ as follows:

$$f_S \in \int I_S \iff \Sigma_S \subset \mathcal{C}_{X_S/S} \left(\iff (f_S, \partial_i f_S) \subset I_S\right).$$

In other words, there is an exact sequence

$$0 \longrightarrow I_S/\int I_S \xrightarrow{d} \Omega^1_{P_S/S} \otimes \mathcal{O}_{\Sigma_S} \longrightarrow \Omega^1_{\Sigma_S/S} \longrightarrow 0 \qquad (14)$$

which is the relative version of (2).

Theorem (3.31) :

Let Σ be a reduced space and let $\Sigma_S \longrightarrow S$ be a deformation of Σ over S. Let I_S be the ideal of Σ_S. Then there is an inclusion

$$I_S/\int I_S \hookrightarrow N_{\Sigma_S}^*$$

and hence an equivalence

$$\Sigma_S \hookrightarrow X_S \text{ admissible } \left(\iff f_S \in \int I_S\right) \iff$$

$$\text{the } \alpha \text{ - map } \alpha_S : N_{\Sigma_S} \longrightarrow \mathcal{O}_{\Sigma_S} \text{ is the zero map.}$$

proof : The (almost) dual of (14) is the exact sequence

$$0 \longrightarrow \Theta_{\Sigma_S} \longrightarrow \Theta_{P_S/S} \otimes \mathcal{O}_{\Sigma_S} \longrightarrow N_{\Sigma_S} \longrightarrow T^1_{\Sigma_S/S} \longrightarrow 0$$

where the group at the right hand side can be interpreted as the first order deformations of the map $\Sigma_S \longrightarrow \mathrm{Spec}(S)$. The dual of this sequence starts with the exact segment:

$$0 \longrightarrow \mathrm{Hom}_{\Sigma_S}(T^1_{\Sigma_S/S}, \mathcal{O}_{\Sigma_S}) \longrightarrow N^*_{\Sigma_S} \longrightarrow \Omega^1_{P_S/S} \otimes \mathcal{O}_{\Sigma_S}$$

We claim that the group at the left hand side is actually zero. Using (0.3) we can conclude:

$$\mathrm{Hom}_{\Sigma}(T^1_{\Sigma_S/S} \otimes \mathcal{O}_{\Sigma}, \mathcal{O}_{\Sigma}) = 0 \quad \Rightarrow \quad \mathrm{Hom}_{\Sigma_S}(T^1_{\Sigma_S/S}, \mathcal{O}_{\Sigma_S}) = 0.$$

But as Σ is reduced by assumption, and $T^1_{\Sigma_S/S} \otimes \mathcal{O}_{\Sigma}$ is a torsion $\mathcal{O}_{\Sigma}$ module, this first Hom is indeed zero. ⊠

This alternative way to express admissibility of $\Sigma_S \hookrightarrow X_S$ also leads to an alternative obstruction theory. For this to work we need an extra condition on Σ.

Lemma (3.32) :

Assume that $T^2_{\Sigma} = 0$.

i) The normal bundle is compatible with restriction, i.e. if $S' \longrightarrow S$ is a surjection of rings and $\Sigma_{S'} \longrightarrow \mathrm{Spec}(S')$ is a deformation, then one has $N_{\Sigma_{S'}} \otimes S = N_{\Sigma_S}$.

ii) The exact sequence (12) gives rise to an exact sequence

$$0 \longrightarrow V \otimes N_{\Sigma_{S'}} \longrightarrow N_{\Sigma_{S'}} \longrightarrow N_{\Sigma_S} \longrightarrow 0$$

and an isomorphism $V \otimes N_{\Sigma_{S'}} \approx V \otimes N_{\Sigma}$ (i.e. N_{Σ_S} is flat).

Outline of proof : It is enough to show this for small extensions. N_{Σ_S} can be interpreted as the space of (embedded) deformations of the *map* $\Sigma \longrightarrow \mathrm{Spec}(S)$ over $k[\varepsilon]/(\varepsilon\ )$. With this interpretation, statement i) is equivalent to the extendability of a family over $S' \times S[\varepsilon]/(\varepsilon^2)$ to a family over $S'[\varepsilon]/(\varepsilon^2)$. This is certainly implied by the condition $T^2_{\Sigma}=0$. Statement ii) now follows from (0.1) (take $i=0, R=P, M=I_S, N=\mathcal{O}_{\Sigma_S}$.) ⊠

Let $\xi_S = \left(\Sigma_S \hookrightarrow X_S\right) \in \mathrm{Def}(\Sigma,X)(S)$ be an admissible deformation over S and $\Sigma_{S'} \in \mathrm{Def}(\Sigma)(S')$ a deformation of Σ over S', lifting Σ_S. We will construct an element $ob(\xi_S,\Sigma_{S'}) \in N\ /I \otimes V$, which maps to the element $Ob(\xi_S, \Sigma_{S'}) \in \Omega^1_\Sigma \otimes V$ of (3.20) via the map (13).
The construction is as follows:

* Over S we know that the α - map $\alpha_S : N_{\Sigma_S} \longrightarrow \mathcal{O}_{\Sigma_S}$ is the zero map by (3.29). Hence, for all $m_S \in N_{\Sigma_S}$ there is a $\gamma_S = \gamma_S(m_S)$ such that $\alpha_S \cdot m_S + \gamma_S \cdot \Delta_S = 0$.
* Now a lift of Δ_S to $\Delta_{S'}$ is given. Take arbitrary lifts of α_S to $\alpha_{S'}$, γ_S to $\gamma_{S'}$ and of m_S to $m_{S'} \in N_{\Sigma_{S'}}$. (For m_S this is possible by (3.32)i).) Let $h_{S'} := \alpha_{S'}.m_{S'} + \gamma_{S'}.\Delta_{S'}$
* Consider now $m \in N_\Sigma$, $m = \overline{m_S}$.

  Claim : the homomorphism $h : N_\Sigma \longrightarrow \mathcal{O}_\Sigma \otimes V$; $m \longmapsto h_{S'}$ gives rise to a well - defined element $ob(\xi_S,\Sigma_{S'}) \in N^*/I \otimes V$.

  proof : One has to check several things. For example, when we choose another lift for $\alpha_{S'}$, the difference is of the form $v\otimes\beta$ for some $v \in V$ and β. The quantity $h_{S'}$ then changes by $v\otimes\beta.m$. But by (3.30) this means that the homomorphism h is changed by an element of $I/\int I$, so the class of h in N^*/I stays the same. We omit the further straightforward checks. ⊠

To get an obstruction element only dependent on ξ_S and not on $\Sigma_{S'}$, we have to divide out N^*/I by a subspace that corresponds to $w(T^1_\Sigma)$ in $\mathrm{Tors}(\Omega^1_\Sigma)$. To put this subspace in a proper setting we introduce a symmetric bilinear form on N_Σ which is of independent interest.

Definition (3.33) ;

Let $\Sigma \hookrightarrow X$ be an admissible diagram, defined by $f \in \int I$. Assume that Σ is reduced and that $T^2_\Sigma = 0$.

The **Hessian** $\mathbf{H} : N_\Sigma \times N_\Sigma \longrightarrow \mathcal{O}_\Sigma$ is a symmetric bilinear form defined by the following four steps.

i) Let n and $m \in N_\Sigma$. This means that for all $r \in \mathcal{R}$ we can solve
$r.n + s(n).\Delta = 0$ and $r.m + s(m).\Delta = 0$ for $s(n)$ and $s(m)$. (Of course, $s(n)$ and $s(m)$ will also depend on $r \in \mathcal{R}$.)

ii) Recall that there is, in general, a pairing $T^1_\Sigma \times T^1_\Sigma \longrightarrow T^2_\Sigma$ .The vanishing of this pairing between (the classes of) n and m just means that one can find a p and t such that for all $r \in \mathcal{R}$ one has:

$$r.p + s(n).m + s(m).n + t.\Delta = 0$$

In particular if $T^2_\Sigma = 0$ (as we assumed) this applies.

iii) Because the α - map of f is the zero map, one can solve the equations $\alpha.n + \gamma(n).\Delta = 0$ and $\alpha.m + \gamma(m).\Delta = 0$ for $\gamma(n)$ and $\gamma(m)$.

iv) Now put $\mathbf{H}(n,m) := \alpha.p + \gamma(n).m + \gamma(m).n$.

Proposition (3.34) :

The Hessian form **H** has the following properties:

i) $\mathbf{H} : N_\Sigma \times N_\Sigma \longrightarrow \mathcal{O}_\Sigma$ is well - defined, i.e. it does not depend on the choices made in the above steps.

ii) For $\vartheta \in \Theta$ one has $\mathbf{H}(n,\vartheta(\Delta)) = - \vartheta \lrcorner \omega.n$.

iii) For ϑ_1 and $\vartheta_2 \in \Theta$ one has $\mathbf{H}(\vartheta_1(\Delta), \vartheta_2(\Delta)) = - \vartheta_1(\vartheta_2(f))$.

iv) By transposition we get a map $\mathbf{h} : N_\Sigma \longrightarrow N_\Sigma^*$.
The composition $N_\Sigma \xrightarrow{\mathbf{h}} N_\Sigma^* \hookrightarrow \Omega^1 \otimes \mathcal{O}_\Sigma$ is equal to the map $-\omega : N_\Sigma \longrightarrow \Omega^1 \otimes \mathcal{O}_\Sigma$ of (3.3).

v) If $f \in I^2$, $f = (h.\Delta).\Delta$ for some matrix h, then $\mathbf{H}(n,m) = -2.h.n.m$.

proof : Statement i) follows by a straightforward check. For example, given f, then the difference $\delta\alpha$ of two choices of α is $\in \mathcal{R}$. This $\delta\alpha$ induces $\delta\gamma$'s such that $\delta\alpha.n + \delta\gamma(n).\Delta = 0$ and $\delta\alpha.m + \delta\gamma(m).\Delta = 0$. Then the induced change $\delta\mathbf{H}$ in **H** is given by $\delta\alpha.p + \delta\gamma(n).m + \delta\gamma(m).n$. But by the definition of p this quantity is in the ideal I, hence **H** in $\mathcal{O}_\Sigma$ is independent of the choice of α. Statement ii) can be seen as follows: by differentiating the relations $r.\Delta = 0$ and $r.n + s(n).\Delta = 0$ with respect to $\vartheta \in \Theta$ we get the expressions $r.\vartheta(\Delta) + \vartheta(r).\Delta = 0$ and $r.\vartheta(n) + s(n).\vartheta(\Delta) + \vartheta(r).n + \vartheta(s(n)).\Delta = 0$. Hence $\vartheta(n)$ can be taken as the p of n and $\vartheta(\Delta)$. From $\alpha.\Delta = f$ we get $\gamma(\vartheta(\Delta)) = \vartheta(\alpha) - \vartheta \lrcorner \omega$. Making the substitutions and using $\vartheta(\alpha.n+\gamma(n).\Delta)=0$ we get ii). Statement iii) follows from ii) and expresses the fact that **H** is an extension of the *second derivative* of f from vector fields to normal vectors. Statement iv) is just another way to express ii). Statement v) follows by direct calculation. ⊠

Corollary (3.35) :

i) $P_\Sigma(\mathcal{A}) = \ker(\mathbf{h}: N_\Sigma \longrightarrow N^*/I)$; $T^1_\Sigma(X) = \ker(\mathbf{h}: T^1_\Sigma \longrightarrow N^*/I)$. where the maps **h** are induced by $\mathbf{h} : N_\Sigma \longrightarrow N^*_\Sigma$.

ii) The map $\varphi : P_\Sigma(\mathcal{A}) \longrightarrow I/\int I$ is injective if and only if the map $\mathbf{h} : N_\Sigma \longrightarrow N^*_\Sigma$ injective.

iii) The obstruction $Ob(\xi_S) \in Tors(\Omega^1_\Sigma)/w(T^1_\Sigma) \otimes V$ lifts to an element $ob(\xi_S) \in Coker(\mathbf{h} : T^1_\Sigma \longrightarrow N^*/I) \otimes V \hookrightarrow Tors(\Omega^1_\Sigma)/w(T^1_\Sigma) \otimes V$.

proof : Statement i) follows from (3.34)iv) together with (3.5) and (3.7). Statement ii) follows from the fact that φ is injective if and only if the map $\omega: N_\Sigma \longrightarrow \Omega^1 \otimes \mathcal{O}_\Sigma$ is injective. Now use (3.34)iv) again. Statement iii) is obtained by studying the dependence of $ob(\xi_S, \Sigma_{S'})$ on the chosen lift $\Sigma_{S'}$. We leave the details to the reader. ⊠

The relation between the Hessian form and the number of D_∞ - points appearing in a generic (admissible) perturbation was first noticed by Siersma (see [Si], remark 4.1) in the case that Σ is a smooth curve germ. Pellikaan (see [Pe 2], pp.27-32) generalized this to the case where Σ is a complete intersection curve and more generally to $f \in I^2$ in the case that Σ is *syzygetic* (see [Pe2] for a definition; space curves are syzygetic). He defined the Hessian form in those cases essentially by formula (3.34)v). T. de Jong in [Jo] introduced for a general germ of a hypersurface (X,p) with a one dimensional singular locus Σ and transverse A_1 - singularities an invariant $VD_\infty(X,p)$ (possibly negative! ; for a D_∞ -singularity, $VD_\infty = 1$ and for the triple point $T_{\infty,\infty,\infty}$, $VD_\infty = -2$) and showed an appropriate continuity statement under admissible deformations of $\Sigma \hookrightarrow X$. We will relate this invariant VD_∞ now to the Hessian form **H**.

Reminder (3.36) :

i) Let $(X,p) \subset (\mathbb{C}^{n+1},0)$ be a germ of hypersurface singularity defined by $f \in \mathcal{O}$ and with one dimensional singular locus Σ and transverse type A_1. Put $\Theta_f := \{\vartheta \in \Theta \mid \vartheta(f)=0\}$ and $\Theta(f) := \Theta_f \otimes \mathcal{O}_\Sigma$. Then the virtual number of D_∞- points, VD_∞ is defined by

$$VD_\infty(X,p) := \dim_{\mathbb{C}}\left(\Theta_{\tilde{\Sigma}} / \Theta(f)\right) - 3.\delta(\Sigma,p).$$

where $\tilde{\Sigma}$ is the normalization of Σ and δ the delta invariant.

ii) Let $Y \subset \mathbb{P}^{n+1}$ be a projective hypersurface of degree d, with a one dimensional singular locus Ξ and transverse type A_1. Then:

$$\sum_{p \in \Sigma} VD_\infty(X,p) = (nd - 2(n+2)).\deg(\Xi) + 4.\,\chi(\mathcal{O}_\Xi)$$

iii) Let $(X,p) \subset (\mathbb{C}^{n+1}, 0)$ be a germ as under i). Then there exists a $Y \subset \mathbb{P}^{n+1}$ as in ii) with a point $y \in Y$ such that:

a. $(Y,y) \approx (X,p)$

b. $Y - \{y\}$ has only A_∞ and D_∞ - singularities.

proof : For definition i) see [Jo], where one also finds the proof of the continuity under deformation of VD_∞. Result ii) is proved in [J-J]. The proof of iii) involves Sard type of arguments and results on I-finite determinacy (see [Pe3]) and will appear elsewhere. We remark that because the total number of VD_∞ on a projective surface in $\mathbb{P}^3$ is even, one has in general to admit D_∞ - points in the compactification. ⊠

Theorem (3.37) :

Let (X,p) be a germ of a hypersurface with one dimensional singular locus Σ and transverse type A_1. Assume that $T^2_\Sigma = 0$ and that Σ is smoothable and syzygetic. Then:

$$VD_\infty(X,p) = \dim_{\mathbb{C}}\left(N_\Sigma^*/\mathbf{h}(N_\Sigma)\right) - \dim_{\mathbb{C}}\left(N^*/I\right) + \dim_{\mathbb{C}}\left(\int I/I^2\right)$$

proof : First we globalize the germ (X,p) to get a $Y \subset \mathbb{P}^{n+1}$ as in (3.36) iii). The compactification of Σ is denoted by Ξ. It is not hard to see the the Hessian form $\mathbf{H} : N_\Sigma \otimes N_\Sigma \longrightarrow \mathcal{O}_\Sigma$ can be globalized to a map $\mathcal{H}_Y : N_\Xi \otimes N_\Xi \longrightarrow \mathcal{O}_\Xi \otimes N_Y$, where N_Y is the normal bundle of Y in $\mathbb{P}^{n+1}$. Now it is checked by calculation that the theorem is true for the A_∞ and the D_∞ - singularity. So by statement ii) of (3.36) it is sufficient to prove that

$$\sum_{p \in \Xi} \dim\left(N_{\Xi,p}^*/\mathbf{h}_p(N_{\Xi,p})\right) - \dim\left(N_{\Xi,p}^*/(I/\int I)_p\right) + \dim\left((I/\int I)_p\right) =$$

$$= (nd - 2.(n+2))\deg(\Xi) + 4.\chi(\mathcal{O}_\Xi)$$

where the index p refers to the local invariant of (Y,p) at that point. The global Hessian gives rise to an injective map

$$h_Y : N_\Xi \longrightarrow N_\Xi^* \otimes N_Y$$

with as cokernel a sky scraper sheaf at p of lenght $N_{\Xi,p}^*/\mathbf{h}_p(N_{\Xi,p})$. Hence the left hand side of formula above is equal to

$$\chi(N_\Xi^* \otimes N_Y) - \chi(N_\Xi) - \chi(N_\Xi^*) + \chi(I/I^2).$$

Furthermore, $\chi(N_\Xi^* \otimes N_Y) = \chi(N_\Xi^*) + 2.n.\deg(Y).\deg(\Xi)$, by Riemann-Roch, because N_Y is a line bundle. So the statement is equivalent to:

$$\chi(I/I^2) - \chi(N_\Xi) = 4.\chi(\mathcal{O}_\Xi) - 2.(n+2).\deg(\Xi). \qquad (*)$$

But this is a statement that only depends on the curve Ξ. By a theorem of Pellikaan (see [Pe2], thm. 4.5) we know that the statement of the theorem is true if we start with $f \in I^2$ and so the theorem holds for any f. (Another line of argument to see (*) is as follows: (*) is true for smooth Ξ; the conditions syzygetic and $T^2_\Sigma = 0$ give that the left hand side is constant under deformation of Ξ; Σ is assumed to be smoothable, so (*) must be true.) ⊠

Remark (3.38) :

It is desirable to have a more conceptual or *local* proof of the above theorem. We do not know whether the theorem is true for non - smoothable curves Σ. But in any case the conditions on Σ are satisfied when X is a germ of a weakly normal surface in $\mathbb{C}^3$.

The folowing corollary is a partial generalization of [Pe 2], thm.1.13.

Corollary (3.39) :

Under the same assumptions as for (3.37) we have:

$$j(f) = c_{I,e}(f) + VD_\infty(X,p) + \dim_{\mathbb{C}} T^1_\Sigma - \dim_{\mathbb{C}}\left(\textstyle\int I/I^2\right)$$

where $j(f) := \dim_{\mathbb{C}}\left(I/J(f)\right)$ and $c_{I,e}(f) := \dim_{\mathbb{C}}\left(\int I + J(f)/J(f)\right)$.

proof : Consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Theta\otimes\mathcal{O}_\Sigma/\Theta_\Sigma & \longrightarrow & N_\Sigma & \longrightarrow & T^1_\Sigma & \longrightarrow & 0 \\ & & \downarrow \lrcorner df & & \downarrow h & & \downarrow h & & \\ 0 & \longrightarrow & I/\int I & \longrightarrow & N^*_\Sigma & \longrightarrow & N^*/I & \longrightarrow & 0 \end{array}$$

The map $\lrcorner df$ is induced by $\vartheta \in \Theta \longmapsto \vartheta \lrcorner df = \vartheta(f)$

The map $h: N_\Sigma \longrightarrow N^*_\Sigma$ is injective and hence the $\lrcorner df$ is injective. Comparing the indices of the vertical maps of the diagram then gives:

$$\dim_{\mathbb{C}}\left(I/\textstyle\int I + J(f)\right) = \dim_{\mathbb{C}}\left(N^*_\Sigma/h(N_\Sigma)\right) - \dim_{\mathbb{C}}\left(N^*/I\right) + \dim_{\mathbb{C}}\left(T^1_\Sigma\right)$$

The exact sequence

$$0 \longrightarrow (\textstyle\int I + J(f))/J(f) \longrightarrow I/J(f) \longrightarrow I/(\textstyle\int I + J(f)) \longrightarrow 0$$

then gives the corollary when we use (3.37). ⊠

We give another noteworthy formule for VD_∞ for a germ (X,p) of a weakly normal surface in $\mathbb{C}^3$. We let $n : \tilde{X} \longrightarrow X$ be the normalization map, and let $\tilde{\Sigma} = n^{-1}(\Sigma)$ be the inverse image of Σ under the normalization map. We let b be the number of irreducible components of X at p.

Theorem (3.40) :

With the notation as above we have:

$$VD_\infty(X,p) = \mu(\tilde{\Sigma}) - 2.\mu(\Sigma) + 2 - b.$$

proof : We compactify X and Σ as in (3.36) iii) to get a Y and Ξ. Let $\deg(Y) = d$, $\deg(\Xi) = e$. Consider the normalization diagram:

$$\begin{array}{ccc} \tilde{\Xi} & \hookrightarrow & \tilde{Y} \\ \downarrow & & \downarrow n \\ \Xi & \hookrightarrow & Y \end{array}$$

where $n: \tilde{Y} \longrightarrow Y$ is the normalization map and $\tilde{\Xi} = n^{-1}(\Xi)$.
Note that because X is weakly normal, the ideal sheaf $\mathcal{D}$ of Ξ in $\mathcal{O}_Y$ is equal to the conductor $\mathcal{H}om_Y(n_*\mathcal{O}_{\tilde{Y}}, \mathcal{O}_Y)$.
From the exact sequence $\mathcal{D} \hookrightarrow \mathcal{O}_Y \twoheadrightarrow \mathcal{O}_\Xi$ we get after applying $\mathcal{H}om_Y(-,\mathcal{O}_Y)$ the exact sequence

$$0 \longrightarrow \mathcal{O}_Y \longrightarrow n_*\mathcal{O}_{\tilde{Y}} \longrightarrow \mathcal{E}xt^1_Y(\mathcal{O}_\Xi, \mathcal{O}_Y) \longrightarrow 0 .$$

But $\mathcal{E}xt^1_Y(\mathcal{O}_\Xi, \mathcal{O}_Y) = \mathcal{E}xt^1_Y(\mathcal{O}_\Xi, \omega_Y) \otimes \omega_Y^{-1} = \omega_\Xi \otimes \omega_Y^{-1}$ because ω_Y is a line bundle and using the adjunction formula. Hence:

$$0 \longrightarrow \mathcal{O}_Y \longrightarrow n_*\mathcal{O}_{\tilde{Y}} \longrightarrow \omega_\Xi \otimes \omega_Y^{-1} \longrightarrow 0$$

The normalization diagram is a pull - back, hence we also get an exact sequence of the form (see [Str], 1.2.3)

$$0 \longrightarrow \mathcal{O}_\Xi \longrightarrow n_*\mathcal{O}_{\tilde{\Xi}} \longrightarrow \omega_\Xi \otimes \omega_Y^{-1} \longrightarrow 0 \qquad (14)$$

So we get $\chi(\mathcal{O}_{\tilde{\Xi}}) = \chi(\mathcal{O}_\Xi) + \chi(\omega_\Xi \otimes \omega_Y^{-1})$

$$\chi(\omega_\Xi \otimes \omega_Y^{-1}) = \chi(\omega_\Xi) + \deg(\mathcal{O}_\Xi \otimes \omega_Y^{-1}).$$

As $\omega_Y \approx \mathcal{O}_Y(d-4)$, we find $\deg(\mathcal{O}_{\Xi} \otimes \omega_Y^{-1}) = e(4-d)$. By Serre duality, $\chi(\mathcal{O}_{\Xi}) = -\chi(\omega_{\Xi})$, hence:

$$\chi(\mathcal{O}_{\tilde{\Xi}}) = e(4-d) \qquad (!)$$

Substitution of this result in formula (3.36) ii) then gives:

$$\sum_{p \in \Xi} VD_{\infty}(Y,p) = 2\big(2.\chi(\mathcal{O}_{\Xi}) - \chi(\mathcal{O}_{\tilde{\Xi}})\big). \qquad (15)$$

One has $2.\chi(\mathcal{O}_{\Xi}) = \mu(\Xi) - \chi_{top}(\Xi)$ (see [Gre], p.149). Remember that $\tilde{\Xi} \longrightarrow \Xi$ is a 2:1 ramified cover, so it is an easy topological matter to relate $\chi(\tilde{\Xi}) - 2.\chi(\Xi)$ to *local* data of the normalization. Using all this we finally find the formula of theorem (3.40). ⊠

§ 4. *Deformations of the Normalization.*

In case we have a hypersurface X with singular locus Σ in codimension 1, we will prove that under certain circumstances there is a natural equivalence

$$\mathrm{Def}(\Sigma, X) \xrightarrow{\approx} \mathrm{Def}(\tilde{X} \longrightarrow X)$$

where $\mathrm{Def}(\tilde{X} \longrightarrow X)$ is the deformation functor of the diagram of the normalization map $\tilde{X} \longrightarrow X$, i.e. the functor of simultaneous normalization of X (see [Bu]). The above equivalence is particularly useful for the study of the deformation theory of normal surface singularities. By projecting such a normal surface singularity into $\mathbb{C}^3$ one gets a hypersurface X together with a curve Σ of double points. By the method of § 3. one can compute the base space of a semi-universal deformation for $\mathrm{Def}(\Sigma, X)$. We will give examples in § 5.

The problem with simultaneous normalization over an infinitesimal basis is that one cannot use the usual construction of integral closure in the total quotient ring to get $\tilde{X}_S$ out of X_S : over $S = \mathrm{spec}(k[\varepsilon]/(\varepsilon^2))$ every element ε/x is integral for $x \in \mathcal{O}_{X_S}$ a non zero divisor. This is reflected in the fact that the natural forgetful transformation $\mathrm{Def}(\tilde{X} \longrightarrow X) \longrightarrow \mathrm{Def}(X)$ is not always injective.

It appears that the missing bit of information to construct $\tilde{X}$ out of X is just the *conductor* $C := \mathcal{H}om_X(\mathcal{O}_{\tilde{X}}, \mathcal{O}_X)$. We can consider $\mathcal{O}_{\tilde{X}}$ as a module over $\mathcal{O}_X$. When we deform $\mathcal{O}_X$ flat over S to an $\mathcal{O}_{X_S}$, it turns out that deforming the $\mathcal{O}_X$-module $\mathcal{O}_{\tilde{X}}$ to an S-flat $\mathcal{O}_{X_S}$ -module $\mathcal{O}_{\tilde{X}_S}$ is equivalent to deforming the conductor C flat to an C_S. However, the conductor C is a very special ideal in $\mathcal{O}_X$: the fact that $\mathcal{O}_{\tilde{X}}$ carries a *ring structure* is equivalent to:

Ring Condition (R.C.)

$$\mathcal{H}om_X(C, C) \xhookrightarrow{\approx} \mathcal{H}om_X(C, \mathcal{O}_X)$$

The last statement makes sense over any basis S, and it turns out that elements of $\mathrm{Def}(\tilde{X} \longrightarrow X)(S)$ correspond to deformations of X and C to X_S and C_S for which C_S still satisfies the corresponding condition (R.C.). To be precise, one has the following theorem:

Theorem (4.1) :

Let $\tilde{X} \longrightarrow X$ be a finite surjective and generically injective mapping. Let Σ be the subspace of X defined by the conductor ideal $C = \mathcal{H}om_X(\mathcal{O}_{\tilde{X}}, \mathcal{O}_X)$. Assume that:

i) $\tilde{X}$ is Cohen - Macaulay

ii) X is Gorenstein

Then there is a natural equivalence of functors

$$\mathrm{Def}(\tilde{X} \longrightarrow X) \longrightarrow \mathrm{Def}(\Sigma \hookrightarrow X, \mathrm{R.C.})$$

Here the second functor is deformations of the diagram $\Sigma \hookrightarrow X$ for which the ideal of Σ_S in X_S satisfies condition R.C.

The next thing to do is to relate (R.C.) to admissibility. For this we need some more conditions on X and Σ.

Theorem (4.2) :

Let $\Sigma \hookrightarrow X$ be an admissible diagram. Assume that;

i) X is a hypersurface

ii) Σ is Cohen-Macaulay of codimension 2.

iii) Σ is reduced.

Let $\Sigma_S \hookrightarrow X_S$ be any deformation of this diagram over S. Then equivalent are:

i) the map $\alpha_S : N_{\Sigma_S} \longrightarrow \mathcal{O}_{\Sigma_S}$ is the zero map (see (3.31)).

ii) the ideal I_S of Σ_S satisfies (R.C.).

iii) the diagram $\Sigma_S \hookrightarrow X_S$ is admissible.

When we combine theorems (4.1) and (4.2) we get the following:

Theorem (4.3) :

Let $\tilde{X} \longrightarrow X$ be a finite, generically injective map. Let Σ be the subspace of X defined by the conductor. Assume that:

i) $\tilde{X}$ is Cohen-Macaulay.

ii) X is a hypersurface.

iii) Σ is reduced.

Then there is a natural equivalence of functors

$$\mathrm{Def}(\tilde{X} \longrightarrow X) \longrightarrow \mathrm{Def}(\Sigma, X).$$

To complete the picture we state one other theorem

Theorem (4.4) :

Under the same conditions as in theorem (4.3) one has that the natural forgetful transformation

$$\mathrm{Def}(\tilde{X} \longrightarrow X) \longrightarrow \mathrm{Def}(\tilde{X})$$

is smooth.

Theorems (4.3) and (4.4) together imply that the base space of the semi-universal deformation of $\tilde{X}$ is, up to a smooth factor, the same as the base space of the functor Def(Σ,X). So the whole complexity of deformations of normal surfaces is reflected in the theory of admissible deformations of weakly normal (i.e. generically transverse A_1) surfaces in $\mathbb{C}^3$.

The rest of this paragraph is devoted to the proofs of the above stated theorems. For notational convenience and clarity of exposition we change from geometric language to algebraic language.

Let R and S rings as in § 0. Let $QR \supset R$ be the total quotient ring of R.

Definition (4.5):

A *fractional ideal* is a finitely generated R - module M such that:

i) $M \subset QR$

ii) M contains a non - zero divisor.

Lemma (4.6) :

i) If $\overline{M}$ is a fractional ideal in $Q\overline{R}$ and M is an S - flat R-module, then M is a fractional ideal in QR.

ii) Let M and N be fractional ideals in QR. Then $\mathrm{Hom}_R(M,N)$ is also a fractional ideal and can be identified with $\{x \in QR \mid x.M \subset N\}$.

proof : Left as an exercise to the reader. We only note that the map from $\mathrm{Hom}_R(M,N)$ to QR is given by: $(\varphi : M \longrightarrow N) \longmapsto \varphi(m)/m$ (m non - zero divisor in M). ⊠

Proposition (4.7):

Let R be a Gorenstein ring over S, i.e. $\omega_{R/S} \approx R$. Then the duality functor $M \longmapsto M^{\vee} := \mathrm{Hom}_R(M,R)$ on the category of R-modules has the following properties:

i) it converts fractional ideals into fractional ideals.

ii) it converts MCM's over S to MCM's over S (see (0.4)).

iii) it is an inclusion reversing involution on the category of fractional MCM's over S

iv) it commutes with specialization for MCM's, i.e. $(\overline{M})^{\vee} = \overline{(M^{\vee})}$

proof : i) follows from (4.6)ii) and ii) follows from the Gorenstein assumption and proposition (0.10)i). The involutivity iii) results from (0.10)ii) , whereas iv) follows from (0.7)iii) (and (0.9)) ⊠

When a fractional MCM happens to be an overring $\tilde{R}$ of the ring R, then its dual module $C = \mathrm{Hom}_R(\tilde{R},R)$ is an ideal in R, called the *conductor* of $\tilde{R}$ over R. This conductor has a special property:

Proposition (4.8) :

Let $\tilde{R} \supset R$ be a fractional MCM over S and let $C \subset R$ be its dual module. Then equivalent are:

i) $\tilde{R}$ is a ring (with ring structure induced from $\tilde{R} \subset QR$)

ii) The ideal C satisfies the *Ring Condition (R.C.)* , i.e. the natural inclusion map

$$\mathrm{Hom}_R(C,C) \hookrightarrow \mathrm{Hom}_R(C,R)$$

is an isomorphism.

proof : ii) $\Rightarrow$ i) : as we have $\tilde{R}=\mathrm{Hom}_R(C,R)$ by (4.7)iii) we see that if $\mathrm{Hom}_R(C,C) \approx \mathrm{Hom}_R(C,R)$ then $\tilde{R}$ gets the ring structure as the endomorphisms of the R - module C.

i) $\Rightarrow$ ii) : for this we need the 'duality lemma for finite maps' (see[Ha], ex. 6.10, p.239) or 'change of rings isomorphism'

$$\mathrm{Hom}_{\tilde{R}}(M,\mathrm{Hom}_R(\tilde{R},N)) \approx \mathrm{Hom}_R(M,N) .$$

(Here M is any finitely generated $\tilde{R}$ - module and N any R - module.) Now it easy to see that the conductor C is also an $\tilde{R}$ - ideal, so we can take $M=C$ and $N=R$ in the above formula to get $\mathrm{Hom}_{\tilde{R}}(C,C)= \mathrm{Hom}_R(C,R)$. But clearly one has $\mathrm{Hom}_R(C,C) \supseteq \mathrm{Hom}_{\tilde{R}}(C,C)$. Combining these last two facts we get $\mathrm{Hom}_R(C,C)=\mathrm{Hom}_R(C,R)$. ⊠

proof of theorem (4.1) : Start with a map $\tilde{X} \longrightarrow X$ as in the statement of the theorem. Consider a deformation X_S over S. Then the category of diagrams $\tilde{X}_S \longrightarrow X_S$ corresponds exactly to the fractional MCM's for the ring $R = \mathcal{O}_{X_S}$ having $\mathcal{O}_{\tilde{X}}$ as special fibre. By (4.7) and (4.8) the duality functor transforms these into diagrams $\Sigma_S \hookrightarrow X_S$ for which the ideal satisfies (R.C.). ⊠

We now turn to the proof of theorem 4.2 . Let P be the local ring of the ambient space, which is regular over the local ring S of the base. We assume that X_S is a hypersurface, so the local ring R of X_S is of the form R = P/(F), where $F \in P$ is a non - zero divisor. Let I be the ideal of Σ_S in the ring R, so the local ring of Σ_S is R/I. As a subspace of the ambient space, Σ_S is given by an ideal I_P in the ring P. By assumption, Σ_S is CM over S of codimension 2. This implies that the equations of Σ_S are of a special form.

Lemma (4.9) :

There exists a free resolution of I_P as a P - module of the following form:

$$0 \longrightarrow P^r \xrightarrow{\ M\ } P^{r+1} \xrightarrow{\ \Delta\ } I_P \longrightarrow 0$$

Here M is a certain $r \times (r+1)$ matrix and the generators Δ_i of I_P (i.e. the components of the map Δ) are given by the the $r \times r$ minors of M.

proof : The resolution of $\bar{I}_P$ over $\bar{P}$ has the form as above, by the theorem of Hilbert-Burch-Schaps (see [Ar 1], pp.16-17.) As I_P is S - flat by assumption, we find a resolution as above over the ring P. ⊠

Because Σ_S is a subspace of X_S we have $F \in I_P$, i.e. we can write :

$$F = \sum_{i=0}^{r} \alpha_i \cdot \Delta_i$$

Proposition (4.10) :

There is a free resolution of I over P of the form:

$$0 \longrightarrow P^{r+1} \xrightarrow{\ \tilde{M}\ } P^{r+1} \xrightarrow{\ \Delta\ } I \longrightarrow 0$$

Here the matrix $\tilde{M}$ is obtained from the matrix M by adjoining the vector $(\alpha_0 , \alpha_1 , \dots , \alpha_r)$ as zeroth column, so $\det(\tilde{M}) = F$.

proof : As we have $R/I = P/I_P$ we get an exact sequence of the form:

$$0 \longrightarrow P.F \longrightarrow I_P \longrightarrow I \longrightarrow 0$$

The result now follows from (4.9) and the following commutative diagram from which one can conclude the exactness of the bottom row.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P^r & \xrightarrow{M} & P^{r+1} & \xrightarrow{\Delta} & I_P & \longrightarrow & 0 \\ & & \downarrow & & \downarrow = & & \downarrow & & \\ 0 & \longrightarrow & P^{r+1} & \xrightarrow{\tilde{M}} & P^{r+1} & \xrightarrow{\Delta} & I & \longrightarrow & 0 \end{array}$$

⊠

Corollary (4.11) :

i) The module I has a 2 - periodic resolution over the ring R of the following form:

$$\ldots \xrightarrow{\Phi} \mathcal{G} \xrightarrow{\Psi} \mathcal{F} \xrightarrow{\Phi} \mathcal{G} \longrightarrow I \longrightarrow 0 .$$

Here $\Phi = \tilde{M}$ mod F and $\Psi = \bigwedge^r \Phi$ is the *Cramer matrix* of Φ, i.e. the matrix having as entries the $r \times r$ - minors of Φ. $\mathcal{F}$ and $\mathcal{G}$ are free R - modules of rank (r+1).

ii) The dual module $\check{I} = \mathrm{Hom}_R(I, R)$ has a 2 - periodic resolution over the ring R of the form:

$$\ldots \xrightarrow{\check{\Phi}} \check{\mathcal{F}} \xrightarrow{\check{\Psi}} \check{\mathcal{G}} \xrightarrow{\check{\Phi}} \check{\mathcal{F}} \longrightarrow \check{I} \longrightarrow 0$$

Here $\check{\Phi}$ is the transpose of the map Φ.

iii) One has $I \approx \mathrm{Coker}(\Phi) \approx \mathrm{Ker}(\Phi) \approx \mathrm{Im}(\Psi)$

and $\check{I} \approx \mathrm{Coker}(\check{\Phi}) \approx \mathrm{Ker}(\check{\Phi}) \approx \mathrm{Im}(\check{\Psi})$.

proof : It is a standard matter to come from the the resolution over the ring P to a resolution over R. (Matrix factorization, see [Ei].) Hence we get i). ii) is obtained by dualizing i) and using iii), which follows from the 2 - periodicity of the complex under i). ⊠

Let $N := \mathrm{Hom}_P(I_P, P/I_P)$ be the normal bundle of Σ_S in the ambient space. In §3. we already encountered the so-called α-map

$$\begin{array}{llll} \alpha : & N & \longrightarrow & R/I \\ & (\varphi: \Delta_i \longmapsto n_i) & \longmapsto & \sum \alpha_i \cdot n_i \end{array}$$

The pivotal result about the α - map is the following.

Theorem (4.12) :

With the notation as above, the following are equivalent:

i) $\mathrm{Hom}_R(I,I) = \mathrm{Hom}_R(I,R)$, i.e. I satisfies (R.C.).

ii) the entries of the matrix Ψ are in I.

iii) the α - map $\alpha : N \longrightarrow R/I$ is the zero map.

proof : By (4.11) iii) , an element $\delta \in \mathrm{Hom}_R(I,R) = I^\vee$ is represented by an element δ' of $\mathcal{G}^\vee$ in $\mathrm{Im}(\Psi^\vee)$. To evalute δ on an element $i \in I$, represent i by an element $i' \in \mathcal{G}$ and let δ' act on i' . As $\delta' \in \mathrm{Im}(\Psi^\vee)$ we see that the ideal generated by the matrix elements of $\Psi^\vee$ (or Ψ) is the ideal generated by the $\delta(i)$, $\delta \in I^\vee$, $i \in I$. Hence i) $\Leftrightarrow$ ii).

Because Σ_S is CM over S of codimension 2, a generating set of N can be obtained by 'perturbing' the matrix M (see [Ar1], p.16-21). To be more precise, let λ be any $r \times (r+1)$ matix with entries in R. Then one has:

$$\bigwedge^r (M + \varepsilon.\lambda) = \bigwedge^r (M) + \varepsilon.\bigwedge^{r-1}(M) \wedge \lambda \quad \mathrm{mod}\ \varepsilon^2 .$$

So λ gives rize to a normal vector $n^\lambda \in N$ corresponding to the homomorphism n^{λ} : $I_P \longrightarrow R/I$

$$\Delta_i \longmapsto (\wedge^{r-1}(M) \wedge \lambda)_i \ .$$

A little calculation then shows that

$$\alpha(n^{\lambda}) = \text{Trace}\ (\tilde{\Psi} \cdot \lambda)$$

where $\tilde{\Psi}$ is the matrix obtained from Ψ by erasing the 0th row. When we let λ run over the elementary matrices e_{ij} , $1 \le i \le r$, $0 \le j \le r$ we get

$$\alpha(n^{e_{ij}}) = \Psi_{ij}$$

and hence the equivalence between ii) and iii). ⊠

Remark (4.13):

Property ii) in (4.12) can be reformulated as a property of the matrix $\tilde{M}$ or Φ and is called the *Rank Condition* in [Ca] and [M-P]: an $(r+1)\times(r+1)$ matrix Φ is said to satisfy the Rank Condition if the ideal generated by the $r\times r$ - minors of Φ is the same as the ideal generated by the $r\times r$ - minors of the matrix obtained from Φ by deleting the first (zeroth) column. Catanese [Ca] also calls this the Rouché - Capelli property. In any case, the abbreviation (R.C) seems extremely appropriate. For a discussion of the equations defining the ring $\text{Hom}_R(I, I)$ we refer to [Ca] and [M-P].

proof of theorem (4.2): By (4.12) we have that (R.C) is equivalent to the condition that the α - map is the zero map. By theorem (3. 31) we have: If $\bar{I}$ is a *reduced* ideal in $\bar{R}$ then

$$\alpha\text{ - map is the zero map} \Leftrightarrow (F, J_F) \subset I \ .$$

Hence, under the assumptions of (4.2) we have indeed :

$$I = I_S \text{ satisfies (R.C)} \Leftrightarrow \alpha = \alpha_S \text{ - map is zero map} \Leftrightarrow$$

$$\Leftrightarrow \Sigma_S \hookrightarrow X_S \text{ is admissible.}$$ ⊠

Remark (4.14) :

Theorem (4.3) states that in case of a finite generically injective map $\tilde{X} \longrightarrow X$ between a Cohen - Macaulay space $\tilde{X}$ and a hypersurface X with a *reduced* conductor we have an equivalence of functors

$$\mathrm{Def}(\tilde{X} \longrightarrow X) \xrightarrow{\approx} \mathrm{Def}(\Sigma, X)$$

where Σ is given the conductor structure.
We have seen in (2.6) an example where $\mathrm{Def}(\Sigma,X) \longrightarrow \mathrm{Def}(X)$ is not injective, although Σ was reduced and X had an isolated singular point. One calculates that in (2.6) the conductor of the normalization map is just the maximal ideal. Hence, by theorem (4.3) this means that (2.6) also affords an example where $\mathrm{Def}(\tilde{X} \longrightarrow X) \longrightarrow \mathrm{Def}(X)$ is not injective, i.e. an example of non- uniqueness of the normalization mapping in deformational context. This can also been seen directly: Consider the normalization mapping:

$$\begin{array}{ccc} X & \longleftarrow & \tilde{X} \\ k[x,y]/(x^3+y^2) & \hookrightarrow & k[t] \\ x & \longmapsto & -t^2 \\ y & \longmapsto & t^3 \end{array}$$

This mapping can be deformed non - trivially over $k[\varepsilon]/(\varepsilon^2)$ without changing the image by $x \longmapsto -t^2 - 2\varepsilon$; $y \longmapsto t^3 + 3\varepsilon t$. This corresponds exactly to the deformation in (2.6). More generally, for the normalization mapping $\tilde{X} \longrightarrow X$ of a curve germ X one has the following result (see [Bu], p. 82):

$$\dim\Big(\ker\Big(\mathrm{Def}(\tilde{X} \longrightarrow X)(k[\varepsilon]/(\varepsilon^2)) \longrightarrow \mathrm{Def}(X)(k[\varepsilon]/(\varepsilon^2))\Big)\Big) = m - r$$

where m is the multiplicity and r the number of branches of X at the special point of X.

Remark (4.15) :

Let I be an MCM ideal in a hypersurface ring R satisfying (R.C) and let $\tilde{R} = \mathrm{Hom}_R(I,I) = I^\vee \supset R$ the ring extension of R belonging to it. As the complex (4.11) i) is 2 - periodic, it is not hard to compute all the higher Ext' s of I. The result is:

$$\begin{array}{ll} * & \mathrm{Hom}_R(I,I) = \tilde{R} \\ * & \mathrm{Ext}_R^{2k+1}(I,I) = N \\ * & \mathrm{Ext}_R^{2k}(I,I) = \tilde{R}/I \end{array} \left. \right\} \quad k = 0, 1, 2, \ldots$$

In fact, taking $\mathrm{Hom}_R(I,-)$ to the exact sequence

$$0 \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow 0$$

we get a long exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(I,I) \longrightarrow \mathrm{Hom}_R(I,R) \longrightarrow \mathrm{Hom}_R(I,R/I) \longrightarrow$$
$$\longrightarrow \mathrm{Ext}_R^1(I,I) \longrightarrow \mathrm{Ext}_R^1(I,R) \longrightarrow \ldots$$

As I is assumed to be MCM (over S) and $R \approx \omega_{R/S}$ we have that $\mathrm{Ext}_R^1(I,R) = 0$. Hence, I satisfies (R.C) $\Leftrightarrow$ $N \approx \mathrm{Ext}_R^1(I,I)$, where $N = \mathrm{Hom}_R(I,R/I)$ is the normal bundle of Σ_S in X_S . Note that this normal bundle is also equal to the normal bundle of Σ_S in the ambient space $\mathrm{Hom}_P(I_P, R/I)$ if the α - map is zero.
The 2 - periodicity gives that $\mathrm{Ext}_R^2(I,I)$ is a quotient of $\mathrm{Hom}_R(I,I)$. One can check that annihilator is precisely I.
Thus we get a Yoneda Ext - pairing

$$\begin{array}{cccccc} & \mathrm{Ext}_R^1(I,I) & \times & \mathrm{Ext}_R^1(I,I) & \longrightarrow & \mathrm{Ext}_R^2(I,I) \\ Y : & N & \times & N & \longrightarrow & \tilde{R}/I \end{array}$$

In general, this pairing is not symmetric. One can proof that the symmetrization Y^+ of Y takes values in R/I and can be identified with the Hessian **H** of § 3 (see (3.32)). We do not have an interpretation of the anti-symmetric part Y^- of Y, although we expect it to contain interesting new information.

In the deformation theory of X together with the module I one encounters natural maps $T_X^i \longrightarrow Ext_X^{i+1}(I,I)$. We only state:

* $T_X^0 \longrightarrow Ext_X^1(I,I)$ is the zero - map.
* $T_X^1 \longrightarrow Ext_X^2(I,I)$ has as kernel $I/(f,J(f))$

Indeed, for $g \in I$ one can lift the module I over the hypersurface with equation $f + \varepsilon.g$, $\varepsilon^2 = 0$. But it requires extra conditions on g that the deformed I satisfies (R.C) or stays admissible (see § 3).

To conclude this section we give a result that implies theorem (4.4).

Proposition (4.16) :

Let $\tilde{X} \longrightarrow X$ be a mapping and $X \subset Y$ an embedding of X in a space Y smooth over the base field. Then :

i) There is a natural transformation of functors

$Def(\tilde{X} \longrightarrow X) \longrightarrow Def(\tilde{X} \longrightarrow Y)$

ii) The natural transformation $Def(\tilde{X} \longrightarrow Y) \longrightarrow Def(\tilde{X})$ is smooth.

iii) If $\tilde{X}$ is Cohen-Macaulay, X is a hypersurface in Y and the map $\tilde{X} \longrightarrow X$ is generically injective, then the transformation in i) is an equivalence of functors.

Sketch of proof : Let $\left(\tilde{X}_S \longrightarrow X_S\right) \in Def(\tilde{X} \longrightarrow X)$. One can extend the inclusion $X \subset Y$ to an inclusion $X_S \subset Y_S$ $(= Y \times Spec(S))$, because all deformations can be realized by embedded deformations. Now the composition $\tilde{X}_S \longrightarrow X_S \subset Y_S$ determines a well- defined element of $Def(\tilde{X} \longrightarrow Y)$. This gives i). Statement ii) follows immediately from the smoothness of Y. For statement iii) we to construct an inverse to transformation i), i.e. an *image functor.* Let $\left(\tilde{X}_S \longrightarrow Y_S\right)$ be an element of $Def(\tilde{X} \longrightarrow Y)$. Let $\tilde{R}$ be the local ring of $\tilde{X}_S$ and P the local ring of Y. Because $\tilde{R}$ is Cohen-Macaulay (over S), it has a presentation as a P - module as the cokernel of a square matrix $\tilde{N}$ (in fact, it is the transpose of the matrix $\tilde{M}$ of (4.10)). Now define X_S to be the hypersurface in Y_S given by the equation $det(\tilde{N}) = 0$. It is now easy to check that $\left(\tilde{X}_S \longrightarrow X_S\right) \in Def(\tilde{X} \longrightarrow X)(S)$. ⊠

Consider a germ $X \subset \mathbb{C}^3$ of a weakly normal surface and let Σ be the reduced singular locus of X. By theorem (4.3) $Def(\Sigma,X)$ is naturally equivalent to $Def(\tilde{X} \longrightarrow X)$, where $n: \tilde{X} \longrightarrow X$ is the normalization map. Furthermore, by theorem (4.4) the natural forgetful transformation $Def(\tilde{X} \longrightarrow X) \longrightarrow Def(\tilde{X})$ is smooth. Consequentely, the space $T^1_{\tilde{X}}$ of first order deformations of $\tilde{X}$ is a quotient of the space $T^1(\Sigma,X)$ of first order admissible deformations (see § 3). So in order to describe $T^1_{\tilde{X}}$ in terms of $T^1(\Sigma,X)$ we have to identify those first order admissible deformations which deform $\tilde{X}$ trivially. Recall that by theorem (3.15) one has $T^1(\Sigma,X) = P_X(\mathcal{A})/(f,J(f))$, where $f \in \mathbb{C}\{x,y,z\}$ is an equation for X.

Theorem (5.1) :

In the situation as above one has:

$$T^1_{\tilde{X}} = T^1(\Sigma,X)/\mathcal{O}_{\tilde{X}}.J(f) \;\left(= P_X(\mathcal{A})/(f, \mathcal{O}_{\tilde{X}}.J(f)) \right)$$

Here $\mathcal{O}_{\tilde{X}}.J(f)$ is the ideal in $\mathcal{O}_{\tilde{X}}$ generated by $J(f)$.

proof : Let (Φ,Ψ) the matrix factorization as in (4.11). So we have $\mathcal{O}_{\tilde{X}} = \mathrm{Coker}(\Phi^\vee)$. If we choose a basis $1 = u_0, u_1, \ldots, u_t$ for $\mathcal{F}^\vee$ we get an embedding $i : \tilde{X} \hookrightarrow \mathrm{Spec}(\mathbb{C}\{x,y,z\} \otimes \mathbb{C}[u_1, u_2, \ldots, u_t])$- $:= Y$. Part of the equations of $\tilde{X} \subset Y$ is given by:

$$\sum_{i=0}^{r} u_i . \Phi^\vee_{ij} = 0 \qquad j = 0,1,2,\ldots,t. \qquad (*)$$

(For a more complete discussion of the equations of $\tilde{X}$ in Y we refer to Catanese [Ca] and Mond & Pellikaan [M-P].) To get $T^1_{\tilde{X}}$ out of $T^1(\Sigma,X)$ we have divide out the action of all the vector fields on Y , i.e. $T^1_{\tilde{X}} = P_X(\mathcal{A})/(f, \Theta_Y(f))$. As an $\mathbb{C}\{x,y,z\}$ - module, Θ_Y is generated by $u_k . \partial/\partial u_l$, $u_k . \partial/\partial x$, $u_k . \partial/\partial y$, $u_k . \partial/\partial z$. Consider the matrix $\Phi^{\vee(kl)}$ ($k = 0, \ldots, t$; $l = 1, \ldots, t$) with entries:

$$\Phi^{\vee(kl)}_{ij} = \Phi^{\vee}_{ij} + \varepsilon.\delta_{ik}.\Phi^{\vee}_{lj}$$

where δ_{ij} is the Kronecker delta. This matrix satisfies (R.C) over the ring $\mathbb{C}[\varepsilon]/(\varepsilon^2)$, as is easily checked. As $\det(\Phi^{\vee(kl)}) = f + \varepsilon.\delta_{kl}.f$, this gives a trivial deformation of X. But by differentiating (*) with respect to $u_k.\partial/\partial u_l$ we see that the effect of this vector field on the embedding $\tilde{X} \subset Y$ is just described by the matrix $\Phi^{\vee(kl)}$. Hence, to get $T^1_{\tilde{X}}$ from $T^1(\Sigma,X)$ we only have to divide out $\mathcal{O}_{\tilde{X}}.J(f)$. ⊠

In general it is not so easy to use this direct description of $\mathcal{O}_{\tilde{X}}.J(f)$. In fact we have another description of $\mathcal{O}_{\tilde{X}}.J(f) \hookrightarrow \mathcal{O}_X$. We can expand the 1 - forms ω_j of (3.1)(5) into a matrix $\omega = (\omega_{ji})$ defined by

$$\partial f/\partial x_j = \sum_{i=0}^{t} \omega_{ji}.\Delta_i \qquad (j = 0, 1, 2, \ldots, n)$$

Theorem (5.2) :

With the notation and the assumptions as above one has that $\mathcal{O}_{\tilde{X}}.J(f)$ is the ideal generated by the entries of the matrix $\omega.\Psi^{\vee}$.

proof : The elements u_m (m=0, 1, 2, . . . , t) of $\mathcal{O}_{\tilde{X}}$ correspond to the homomorphisms $[u_m]: \Delta_i \longmapsto \Psi^{\vee}_{im}$ of $\mathrm{Hom}_X(I,I) = \mathcal{O}_{\tilde{X}}$.
So $[u_m \partial f/\partial x_k]: \Delta_i \longmapsto \sum \omega_{kl}.\Delta_l.\Psi^{\vee}_{im}$. As we have relations of the form $\Psi^{\vee}_{im}.\Delta_l = \Psi^{\vee}_{lm}.\Delta_i$ (modulo f) we see that the homomorphism $[u_m.\partial f/\partial x_k]$ corresponds to multiplication by $\sum \omega_{kl}.\Psi^{\vee}_{lm} \in \mathcal{O}_X$. ⊠

To compute $T^1_{\tilde{X}}$ we can use any X which has $\tilde{X}$ as normalization. We will give some examples.

Examples (5.3):

1) $f = z^2 - y^2(y + x^k)$. This is the $J_{k,\infty}$ - singularity (see [Si]). The normalization $\tilde{X}$ is smooth, and the ideal $P_X(\mathcal{A}) = (x^k y, z, y^2)$. The matrix factorization of f is given by:

$$\Phi^\vee = \begin{pmatrix} z & y(y+x^k) \\ y & z \end{pmatrix} ; \quad \Psi^\vee = \begin{pmatrix} z & -y(y+x^k) \\ -y & z \end{pmatrix}$$

Furthermore, one can take for the ω - matrix the following:

$$\omega = \begin{pmatrix} 0 & k.x^{k-1}y \\ 0 & y(3y+2x^k) \\ 2 & 0 \end{pmatrix} , \text{ so } \omega.\Psi^\vee = \begin{pmatrix} -k.x^{k-1}y^2 & k.x^{k-1}yz \\ -y(3y+2x^k) & z(3y+2x^k) \\ 2.z & -2.y(y+x^k) \end{pmatrix}$$

So indeed $\mathcal{O}_{\tilde{X}}.J(f) = (y^2, x^k y, z)$ and thus $T^1_{\tilde{X}} = 0$.

2) $f = xz^2 - y^2(y+x^k)$. This is the $Q_{k,\infty}$ - singularity (see [Si]).
We take

$$\Phi^\vee = \begin{pmatrix} xz & y(y+x^k) \\ y & z \end{pmatrix}$$

Because $\mathcal{O}_{\tilde{X}} \approx \mathrm{Coker}(\Phi^\vee)$, we see that $\mathcal{O}_{\tilde{X}}$ is generated as $\mathcal{O}_X$ - module by 1 and $u := xz/y$. The equations of $\tilde{X}$ in $\mathbb{C}^4$ are :

$$u^2 = x.(y+x^k) \quad ; \quad uy = xz \quad ; \quad uz = y(y+x^k) \quad .$$

The inverse image of the singular locus under the normalization map is given by $u^2 = x^{k+1}$; $y = 0$; $z = 0$, and so is an A_k - singularity. The coordinate transformation $u' = u$; $x' = x$; $y' = y+x^k$; $z' = z + ux^{k-1}$ transforms the equations into $(u')^2 = x'.y'$; $u'.y' = x'.z'$; $u'.z' = (y')^2$. Hence, $\tilde{X}$ is isomorphic to the cone over the rational normal curve of degree 3. It is well known that dim $T^1_{\tilde{X}} = 2$ (see [Pi]). One has $(\mathcal{O}_{\tilde{X}}.J(f),f) = (xy^2, xz, z^2, 3y^2 + 2x^k y)$ and $P_X(\mathcal{A}) = (y^2, yz, xz, x^k y)$. (Computations left to the reader). Hence $T^1_{\tilde{X}}$ is represented by the classes of y^2 and yz. We leave it to the reader to do more examples, e.g. one could try $f = xyz + y^{p+3} + z^{q+3}$.

3) $f = (yz)^2 + (zx)^2 + (xy)^2$. This is a continuation of example (3.16)4) and (3.24). Here one has $\tilde{X} = \mathrm{Cone}(\,|\mathcal{O}(4)| : \mathbb{P}^1 \hookrightarrow \mathbb{P}^4\,)$. By Pinkham, (see [Pi]), dim $T^1_{\tilde{X}} = 4$. We already know from (3.16) that:

$$P_X(\mathcal{A}) = (y^2z, yz^2, z^2x, zx^2, x^2y, xy^2, xyz) = m^3 \cap I.$$

One calculates that $(\mathcal{O}_{\tilde{X}}.J(f),f) = m^4 \cap I + J(f)$. A basis for $T^1_{\tilde{X}}$ is represented by $\{xyz,\ y^2z + yz^2,\ z^2x + zx^2,\ x^2y + xy^2\}$.

Let $\tilde{X}$ be a germ of a normal surface singularity. Consider a *smoothing* of $\tilde{X}$, i.e. a deformation $\tilde{X}_S \longrightarrow S$ such that for general $s \in S$ the fibre $\tilde{X}_s$ is smooth. Let S' be the component of the semi--universal deformation on which this smoothing occurs. A theorem of J. Wahl (proved under a condition of globalizability, which Looijenga (see [Lo]) proved to be always fulfilled) relates the dimension of S' to the topology of $\tilde{X}_s$. To formulate this result, let $\pi: Y \longrightarrow \tilde{X}$ be the minimal resolution of $\tilde{X}$, let E be the exceptional divisor and $p_g = \dim(R^1\pi_* \mathcal{O}_Y)$ the geometric genus.

Theorem (J. Wahl, [Wa2], 3.13 c)

$$\dim(S') = \left(\dim H^1(\Theta_Y) - 14 p_g - 2.\chi(E)\right) + 2.\chi(\tilde{X}_s)$$

where χ is the topological Euler characteristic. ⊠

Note that the term in the big brackets only depends on $\tilde{X}$.

Now let $X \subset \mathbb{C}^3$ a germ of a weakly normal surface and let Σ be its reduced singular locus. Corresponding to the notion of a smoothing of the normalization $\tilde{X}$ there is the notion of a *disentanglement* of (Σ,X).

Definition (5.6):

A *disentanglement* of $\Sigma \hookrightarrow X$ is an admissible deformation $\Sigma_S \hookrightarrow X_S$ over a basis S such that for a general $s \in S$ the fibre X_s has only A_∞, D_∞ and $T_{\infty,\infty,\infty}$ - singularities (ordinary double curve, ordinary pinch point, ordinary triple point, c.f. 3.17).

Clearly, the normalization $\tilde{X}_S$ of a disentanglement of X is a smoothing of $\tilde{X}$. We want to compute $\chi(\tilde{X}_s)$ in terms of invariants of X. We choose a Milnor representative for X_S (see [Si]). Let F be the Milnor fibre of X. We can compare F with X and with X_s.

Lemma (5.7) :

With the notations as above we have for a disentanglement:

i) $\chi(\tilde{X}_S) = \chi(X_S) + \chi(\Sigma_S) - D + T$

ii) $\chi(F) = \chi(X_S) - \chi(\Sigma_S) + 2D$

iii) $\chi(\Sigma_S) = 2T - \mu(\Sigma) + 1$

where D is the number of D_∞ and T the number of $T_{\infty,\infty,\infty}$ - points in the disentanglement.

proof : This is a simple excercise in topology. For i) one has to realize that $\tilde{X}_S \longrightarrow X_S$ is 1 - 1 except over Σ where it is 2 - 1 except over the pinch points and the triple points. For ii) one compares F with X_S. As one knows the local structure of the degeneration this is easy. For iii) one has to use the fact that Σ_S is a flat family over S and in such a family the jump in topology is the jump in Milnor numbers (see [B-G]). ⊠

Corollary (5.8) :

i) $$\chi(\tilde{X}_S) = \big(j(f) - 2.VD_\infty(X) - \mu(\Sigma) + 2 \big) - T$$

ii) dim(S') + 2T is an invariant of X and does not depend on the particular disentanglement chosen.

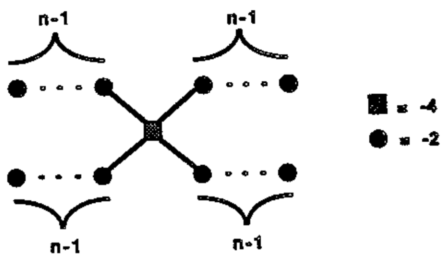
proof : By [Jo], thm.3.2 one has $\chi(F) = j(f) + VD_\infty + \mu(\Sigma)$. Furthermore, one has $VD_\infty = D - 2T$. Combining this with lemma (5.7) we get i). Statement ii) now follows from i) together with the quoted formula of J.Wahl. ⊠

Example (5.9) :

1) X of type D_∞ : $\tilde{X}$ is smooth, hence $\chi(\tilde{X}_S) = 1$. One has $j(f) = 1$, $VD_\infty = 1$ and $\mu(\Sigma) = 0$.

2) $f = (yz)^2 + (zx)^2 + (xy)^2$, $X = \{f = 0\}$. Here the normalization $\tilde{X}$ is Cone($|\mathcal{O}(4)| : \mathbb{P}^1 \hookrightarrow \mathbb{P}^4$). The base space of $\tilde{X}$ has two smoothing components, one of dimension 1 and one of dimension 3 (see [Pi]). One calculates $j(f) = 10$, $VD_\infty = 4$, $\mu(\Sigma) = 2$. For the one dimensional smoothing component we have T = 1 , for the three dimensional component one has T = 0.

The rest of this paragraph is devoted to the study of a particular class of normal surface singularities: the rational quadruple points. We will determine the base space of the semi-universal deformation of such a singularity. The answer turns out to be unexpectedly simple: the isomorphism type of the base space of a rational quadruple point is completely determined by two numbers, s and n. The base space then is isomorphic to $S \times B(n)$, where S is a smooth germ of dimension s and where B(n) is the space defined by the set of equations (6.14). A rational quadruple point with the following star shaped resolution graph



has the factor B(n) in its base space. We call such a singularity an n-star. At the moment of writing, we have been unable to prove most of the following properties of the space B(n). It should not be too hard to settle the following

Conjecture (6.1) :

The space B(n) has the following properties:

1) $\mathrm{Embdim}(B(n)) = 5n - 1$.

2) B(n) has n+1 irreducible components Y_k , $k=0, 1, \ldots, n$ with $\dim(Y_k) = 2n-1+2k$.

3) $\mathrm{Mult}(Y_k) = \binom{n}{k}$, so only Y_0 and Y_n are smooth.

4) Y_k has a smooth normalization.

(Property 1) is trivial and only included for sake of completeness.) We know that (6.1) is true for n = 1 and n = 2 and we know that B(n) has, besides Y_0 and Y_n, at least n-1 other components.

In general there are several approaches to find the semi-universal deformation of a (normal surface) singularity $\tilde{X}$. In the first place there is the *direct method* : one starts with the set of equations defining $\tilde{X}$ as embedded in some high $\mathbb{C}^N$ and then one just computes. For this to work in practice the equations must have a sufficiently strong structure. For example rational triple points (see [Tj 2]) (Cohen-Macaulay codimension 2), the cone over the rational normal curve of degree n (see [Pi]), n lines in $\mathbb{C}^n$ etc, can be handled in this way. It seems however that the equations for the rational quadruple points are not known sufficiently well to compute the base spaces for them in this way. Secondly, there is the method of *(partial) resolutions.* Here one starts with a (partial) resolution Y of $\tilde{X}$ and then studies the deformation theory of Y (which is usually much simper) and finally one tries to blow down the deformed Y to get a deformation of $\tilde{X}$. This methods works quite well to get information on the components of the base space for rational singularities. For example, all deformations of a resolution of $\tilde{X}$ can be blown down and give rise to the so-called *Artin component* of (the base space of) $\tilde{X}$ (see [Wa1]). Recently, Kollar and Shepherd-Barron [K-S] developed a method by which one can, for instance, determine the number of components in the base space of a cyclic quotient singularity. (From their approach it is also clear that the n-star singularity has (at least) n+1 components in its base space.) However, the list of resolution graphs of rational quadruple points is quite long and contains many 'exceptional' graphs, so this method seems to be quite involved. Furthermore, it does not lead really to equations for the base spaces.

We propose to use a different method: the method of *projections.* Here one starts with $\tilde{X}$ embedded in some high $\mathbb{C}^N$ and then we project $\tilde{X}$ generically into $\mathbb{C}^3$. The image X then will have a curve Σ as double locus. By the theorems of §4 the base space of admissible deformations of $\Sigma \hookrightarrow X$ is up to a smooth factor the same as the base space of $\tilde{X}$. Now essentially because Σ is Cohen-Macaulay of codimension 2 and

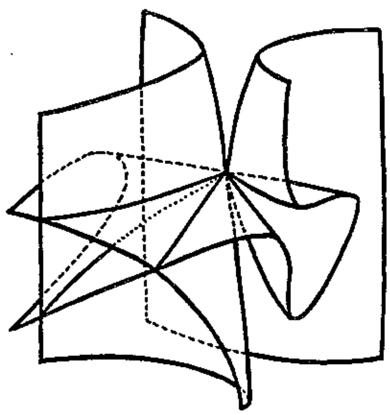
X is given by *one* equation, this is much more 'computable' then working directly with the equations for $\tilde{X}$. At first sight it seems that this method has two serious drawbacks. In the first place one has to choose a *generic* projection (to get an ordinary double curve) and naturally given projections usually are not generic. In the second place it is quite *hard* to find the explicit equation for X. For rational triple points it is already a lot of work to write down explicit equations for X corresponding to the different resolution graphs and for quadruple points it becomes quite hopeless. We only give one example of our (incomplete) list. (It appears that it is convenient to use the theory of *limits* (see [Str]) to obtain equations for singularities that come in series.)

Example (6.2) :

Equation :

$$f = (x-y).\left((x+y).(z^2+xy^2) + (x-y)^{k+1}.y^2\right) + z^1.(z^2+xy^2)^2$$

Qualitative picture of $X_{\mathbb{R}} := \{(x,y,z) \in \mathbb{R}^3 \mid f(x,y,z) = 0\}$:



Resolution graph of the normalization $\tilde{X}$:

So it seems that we are stuck again. But now it turns out that the first mentioned drawback can be turned into an advantage: every singularity has many generic projections with essentially distinct curves Σ. It turns out that *all* curves Σ that could *a priori* occur as double curves of projections of rational quadruple points actually do occur in projections of the special n-star singularities mentioned above. Because $Def(\Sigma,X)$ mainly depends on the curve (in the sense of (3.27) and (3.28)) we get that the base space of any rational quadruple point is the same up to a smooth factor to the base space of a certain n-star singularity. So one has to compute only the base space for the n-star singularity and for this one can use a particular nice set of coordinates and in this way we find the equations for the spaces B(n).

We start with some general numerical relations related to a generic projection.

Lemma (6.3) :

Let $\tilde{X} \subset \mathbb{C}^N$ be a ((multi-) germ of a) normal surface singularity, where $N = \mathrm{Embdim}(\tilde{X})$ is the embedding dimension of $\tilde{X}$. Let $L: \mathbb{C}^N \longrightarrow \mathbb{C}^3$ be a generic linear projection and let $X = L(\tilde{X}) \subset \mathbb{C}^3$ its image. Let Σ be the reduced singular locus of X and let $\tilde{H}$ and H be the generic hyperplane sections of $\tilde{X}$ and X respectively. Then one has:

i) $m := \mathrm{Mult}(\tilde{X}) = \mathrm{Mult}(X) = \mathrm{Mult}(\tilde{H}) = \mathrm{Mult}(H) \geq N-1$.

ii) $\mathrm{Mult}(\Sigma) = \delta(H) - \delta(\tilde{H})$.

iii) $\delta(H) \geq m.(m-1)/2$; $\delta(\tilde{H}) \geq m-1$.

iv) $\mathrm{type}(\Sigma) \geq N-3$.

proof *:* i) is obvious because we have a linear projection. The inequality expresses the minimality of the embedding of $\tilde{X}$ in $\mathbb{C}^N$. Statement ii) follows by moving the hyperplane H away from the special point. We then get as intersection with X a curve with $\mathrm{Mult}(\Sigma)$ ordinary double points. But the jump in δ in a family of curves is equal to the δ of the special fibre of the normalization of the family (see [L-L-T]), so in this case is equal to $\delta(\tilde{H})$. Statement iii) is a generality: given the embedding dimension and the multiplicity of the curve, one has a lower bound for its δ-invariant, which is in the stated cases as above. (exercise). Statement iv) follows the following: Σ is Cohen-Macaulay of codimension 2, so the equations for Σ are obtained as the maximal minors of an $t\times(t+1)$ matrix. Then type(Σ) = t. As in (5.1) this gives us an embedding of $\tilde{X}$ into a smooth space of dimension 3+t, hence $N \leq t+3$. ⊠

Lemma (6.4) :

If $\tilde{X}$ is a germ of a rational surface singularity, then all the inequalities of (6.3) are in fact equalities.

proof : This lemma is a reflection of the strong minimality properties enjoyed by rational singularities. For the fact that N = m + 1 we refer to [Ar2]. For the statement that $\delta(\tilde{H})=m-1$ see([Str], 4.1.13). As now $\tilde{H}$ is a curve with minimal δ-invariant, the same is true for a generic projection H, hence $\delta(H) = m.(m-1)/2$ (we omit the proof). Then Mult(Σ) = (m-1).(m-2)/2. This can also be seen directly: by a result of Karras (see [Ka]) every rational m-tuple point has a normally flat deformation to the cone over the rational normal curve of degree m. When we project this cone to $\mathbb{C}^3$ we get a cone over a rational curve in $\mathbb{P}^2$ of degree m. Such a curve has (m-1).(m-2)/2 double points. As clearly the multiplicity of Σ does not change under this deformation, Mult(Σ) has to have this value for all rational m-tuple points. The statement about the type can be seen as follows: because Σ is Cohen--Macaulay, the sub-scheme of $\mathbb{C}^2$ given by $\Sigma \cap H$ has length (m-1).(m-2)/2 and by (6.3) type($\Sigma\cap H$)$\geq$m-2. From these fact alone it already follows that the ideal of $\Sigma\cap H$ is the ideal $\mathfrak{m}^{m-1}$, where $\mathfrak{m}$ is the maximal ideal of $\mathbb{C}\{y,z\} = \mathcal{O}_{\mathbb{C}^2,0}$. Hence indeed type(Σ)=type($\Sigma\cap H$) = m-2. ⊠

Corollary (6.5) :

$\tilde{X}$ rational triple point $\Rightarrow$ Σ is smooth, i.e. X is a line singularity.

$\tilde{X}$ rational quadruple point $\Rightarrow$ Σ has multiplicity 3 and type 2.

proof: Immediate from (6.4). ⊠

Lemma / Definition (6.6) :

Let Σ be a Cohen-Macaulay curve germ of multiplicity 3 and type 2. Then the equations for Σ can be obtained as the 2×2-minors of the following matrix:

$$M = \begin{pmatrix} y & z+a & b \\ c & y+d & z \end{pmatrix}$$

Here a, b, c and d are functions only depending on x. We define the λ-invariant of such a curve as:

$$\lambda(\Sigma) := \min(\mathrm{ord}(a), \mathrm{ord}(b), \mathrm{ord}(c), \mathrm{ord}(d))$$

Conversely, if $\lambda(\Sigma) \geq 1$, then the minors of the above matrix do define a Cohen-Macaulay curve germ of multiplicity 3 and type 2.

proof : Choose a generic projection of Σ on a line with coordinate x. Then Σ can be considered as the total space of a flat deformation of Σ intersected with x = 0. This sub-scheme of $\mathbb{C}^2$ is defined by $m^2 = (y^2, yz, z^2)$. As these equations can be obtained from the matrix as above (with a=b=c=d=0), we find the indicated form for the equations of Σ. (A similar bigger matrix can be written down for the curves Σ appearing as double locus of a rational m-tuple point with m≥5.) ⊠

Remark (6.7) :

Curves of multiplicity 3 can be classified and J. Stevens has sent us the complete list. However, it turns out to possible to pursue our arguments without going into the fine structure of this classification.

Proposition (6.8) :

Let Σ and coordinates x, y, z as in (5.15). Let $I = (\Delta_1, \Delta_2, \Delta_3)$ the ideal of Σ defined by the minors of the matrix M. Consider the function

$$\Phi := \det(\tilde{M}) \in \mathbb{C}\{x\}[y,z] \; ; \qquad \tilde{M} := \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ y & z+a & b \\ c & y+d & z \end{pmatrix}$$

where $(\alpha_1, \alpha_2, \alpha_3) := x^{-\lambda}.(dy-cz, ad-bc, az-by)$, $\lambda = \lambda(\Sigma)$.

Then Φ has the following properties:

i) $\Phi \in \int I$.

ii) $\mathrm{Mult}(\Phi) = 3$; $\deg_{(y,z)}(\Phi) = 3$; $\Phi(0,y,z) \neq 0$.

iii) Consider a 3x3-matrix h with entries in $\mathbb{C}\{x,y,z\}$ with *generic constant part* h_0. Then the space X defined by $\Phi + h.\Delta.\Delta = 0$ has precisely Σ as singular locus, has a smooth normalization $\tilde{X}$ and the inverse image of $X \cap \{x=0\}$ on $\tilde{X}$ is a smooth curve.

proof : Let us first indicate the geometrical significance of a function Φ having properties i) and ii). The intersection of Σ with the plane $x=c$, $c \neq 0$ consists of three distinct points in the (y,z)-plane. Multiplying together the three linear factors describing the lines through the three pairs of points we get a polynomial Φ of degree 3 in y and z with coefficients depending on x. A direct computation then shows that Φ can be written as the above determinant. The Cramer matrix $\tilde{N}$ of 2x2-minors of $\tilde{M}$ is seen be to equal to

$$\tilde{N} = \begin{pmatrix} \Delta_1 & \alpha\Delta_2 + \beta\Delta_3 & \alpha\Delta_1 + \beta\Delta_2 \\ \Delta_2 & \gamma\Delta_1 + \delta\Delta_2 & \alpha\Delta_2 + \beta\Delta_3 \\ \Delta_3 & \gamma\Delta_2 + \delta\Delta_3 & \gamma\Delta_1 + \delta\Delta_2 \end{pmatrix}$$

(where $(\alpha, \beta, \gamma, \delta) = -x^{-\lambda}.(a,b,c,d)$), which shows that the matrix $\tilde{M}$ satisfies the rank condition, so indeed $\Phi \in \int I$. (see (3.31) and (4.12)). (We thank J. Stevens for pointing out to us that our Φ is equal to $(\Delta_1 . \Delta_3 - \Delta_2^2)/x^{\lambda}$.)

Now we turn to statement iii) of the proposition. The curve $X\cap\{x=0\}$ has an equation of the form $\Phi(0,y,z) + G(y,z)$, where G starts with a *generic* quartic in y and z, because of the genericity of h_0. From this it follows that $\delta(X\cap\{x=0\})= 3$. For small values of c the curve $X\cap\{x = c\}$ has three ordinary double points, hence the $X\cap\{x = c\}$ is a δ-constant family of plane curves. Consequently, X has precisely Σ as singular locus, $\tilde{X}$ is smooth and the inverse image of $X\cap\{x=0\}$ on $\tilde{X}$ is smooth. ⊠

Corollary (6.9) :

In the above situation we have $\int I/I^2 \approx \mathbb{C}[x]/x^{\lambda}$, and a $\mathbb{C}$-basis for $\int I/I^2$ is given by Φ, $x.\Phi$, . . . , $x^{\lambda-1}.\Phi$.

proof : This can be checked by a direct calculation, but it is much nicer to apply here a beautiful theorem of D. Mond & R. Pellikaan (see [M-P], thm.4.4) which implies that for a weakly normal surface X in $\mathbb{C}^3$ with singular locus Σ and with a *Gorenstein* normalization $\tilde{X}$ the module $\int I/I^2$ is cyclic with generator the equation F of X and as annihilator the $(t-1)x(t-1)$-minors of the $(t+1)\times(t+1)$ matrix of the matrix factorization of F (as in (4.10)). In our case $\tilde{X}$ is smooth by (6.8)iii) and t=2, so the annihilator of $\int I/I^2$ is the ideal of of the entries of the matrix $\tilde{M}$ of (6.8), which is the ideal (y,z,x^{λ}). ⊠

Now we have sufficiently detailed information about the structure of the set of all weakly normal surfaces which have a curve Σ of multiplicity three (and type 2): they all have an equation of the form

$$F_{p,h}(\Sigma): = x^{p}.\Phi + (h.\Delta).\Delta$$

where Φ is as in (6.8) , h is a 3x3-matrix with entries in $\mathbb{C}\{x,y,z\}$. We let $X_{p,h}(\Sigma)$ be the surface germ defined by $F_{p,h}(\Sigma) = 0$. Note that by (6.9) $F_{p,h}(\Sigma) \in I^2$ if $p \geq \lambda(\Sigma)$.

Lemma (6.10) :

With the notation as above, let h be a matrix with generic constant part h_0. Then the tangent cone of the surface $X_{p,h}(\Sigma)$ is the cone over a curve $C \subset \mathbb{P}^2$, which has the following structure:

Case A: $\lambda(\Sigma) \geq 2$, $p \geq 2$; C consists of four distinct lines, all passing through a single common point.

Case B: $\lambda(\Sigma) \geq 2$, $p = 1$; C is an irreducible rational quartic curve with a unique singular point of type D_4, D_5 or E_6.

Case C: $\lambda(\Sigma) = 1$, $p = 1$; C is an irreducible rational quartic curve with one (A_5), two ($A_3 + A_1$) or three ($3.A_1$) singular points.

proof : If $p \geq 2$ then the tangent cone of $X_{p,h}(\Sigma)$ is determined by the term $(h.\Delta).\Delta$, because Φ has multiplicity 3. If $\lambda(\Sigma) \geq 2$, then the lowest order terms in the matrix M of (6.6) are the y and z, so for generic h we get as tangent cone a general quartic in y and z, which settles case A. If $p = 1$ and $\lambda(\Sigma) \geq 2$, then the lowest order term of $F_{p,h}$ contains also a term $x.\Phi$. Corresponding to the cases that $\Phi(0,y,z)$ is equivalent to $y^3 + z^3$, $y^2.z$, y^3 we then find a D_4, D_5 or an E_6 on C, which settles case B. The remaining case is $\lambda(\Sigma) = 1$, which is most involved. If we replace a, b, c, d in the matrix M of (6.6) by their linear parts, the minors of it define still a (possibly non-reduced) curve of multiplicity 3, which is the cone over a subscheme of $\mathbb{P}^2$ of length 3. One can check that an irreducible quartic C which contains such a subscheme in its singular locus has to have a total δ equal to 3, hence is rational. For generic h only the indicated cases do occur. ⊠

Theorem (6.11) :

For generic h and $1 \leq p \leq \lambda(\Sigma)$ the surface $X_{p,h}(\Sigma)$ has as normalization a p-star singularity.

proof : We blow up $\mathbb{C}^3$ at the origin. Let Σ' and X' be the strict transforms of Σ and $X_{p,h}(\Sigma)$. X' will have the tangent cone of $X_{p,h}(\Sigma)$ as exceptional divisor. If $p \geq 2$, then $\lambda(\Sigma) \geq 2$, so Σ' will still be a curve germ of multiplicity 3, and $\lambda(\Sigma')=\lambda(\Sigma)-1$, as one easily sees from blowing up the matrix M of (6.6). Also, by (6.10), the exceptional divisor of X' will consist of four lines through a point, which is also the singular point of Σ'. Around this point the surface X' will have a singularity of type $X_{p-1,h'}(\Sigma')$, as follows by looking at the equation $F_{p,h}$ in the x-chart. Because the tangent cone is reduced, X' will be smooth apart from this singularity. As only the constant part of h enters in the genericity assumption for (6.10) and the constant part of h' is the same as that of h, we can thus inductively go further with blowing up. After p-1 blow ups we have introduced four chains of rational curves of lenght p-1 and we are left with a singularity of type $X_{1,h''}(\Sigma'')$. Now there are two cases: $\lambda(\Sigma'') \geq 2$ and $\lambda(\Sigma'')=1$. These correspond to the cases B and C of (6.10). In each of these cases the tangent cone of $X_{1,h''}(\Sigma'')$ is an irreducible rational quartic curve. In the first case we find after one further blow up still a unique special point of type $X_{0,h'''}(\Sigma''')$, which has by (6.8) a *smooth* normalization (and the inverse inverse image of the quartic is also smooth). In the second case we get after blowing up $X_{1,h''}(\Sigma'')$ a surface X''' with singular locus Σ''' which can have one, two or three disjoint parts. We claim that X''' again has a smooth normalization and that the inverse image of the quartic is also smooth. This can be seen by applying the same idea as in the proof of (5.17)iii): around a part of Σ''' the germ of X''' can be considered as the total space of a family of curves with as special fibre the (germ of the) exceptional quartic. It is not hard to see that this is a family with constant δ (equal to 1, 2 or 3), which proves the claim. Our conclusion is that $X_{p,h}(\Sigma)$ for generic h has as normalization a singularity which has as resolution graph the graph of the p-star singularity. By keeping track of the order of vanishing of the function x along all exceptional curves, one can compute all the self- intersections and they are as for the p-star singularty. ⊠

Corollary (6.12) :

Let X be weakly normal surface in $\mathbb{C}^3$ with (reduced) singular locus Σ. Assume that Σ has multiplicity 3 and type 2. Let I be the ideal of Σ and f=0 the equation of X. Then the base space of the semi-universal deformation of the normalization $\tilde{X}$ of X is up to smooth factors the same as the base of the p-star singularity, where $p = \dim\left(\int I/(I^2 + (f))\right)$.

proof : As one has $\int I/I^2 \approx \mathbb{C}[x]/(x^\lambda)$ by (6.9), there are only a finite number of possibilities for $\int I/(I^2 + (f))$, which by (6.11) are all realized by projections of p-star singularities. Now apply (3.28), (4.3) and (4.4).

⊠

To complete the picture, we compute the hull of Def(Σ,X) for a particular nice projection of the n-star singularity.
Let the curve Σ_n be defined by the ideal

$$I = (\Delta_1, \Delta_2, \Delta_3) = (M_2.M_3,\ M_3.M_1,\ M_1.M_2)$$

where $M_i = L_i(y,z) + x^n$ and L_i are linear forms with $L_1 + L_2 + L_3 = 0$.

Lemma (6.13) :

A basis for T^1_Σ is given by the classes of the normal vectors $x^q.A_1$, $x^q.A_2$, $x^q.A_3$ (q=0,1, . . . , n-1) and $x^q.B$ (q=0,1, . . . , n-2) where the A_i and B are

$$\begin{array}{lll} A_1 : (\Delta_1,\Delta_2,\Delta_3) & \longmapsto & (M_2 - M_3,\ 0,\ 0) \\ A_2 : \quad ,, & \longmapsto & (0,\ M_3 - M_1,\ 0) \\ A_3 : \quad ,, & \longmapsto & (0,\ 0,\ M_1 - M_2) \\ B : \quad ,, & \longmapsto & (M_2+M_3,\ M_3+M_1,\ M_1+M_2) \end{array}$$

A basis of Tors(Ω^1_Σ) is given by the classes of the differentials

$$\left.\begin{array}{l} 3x^q\ M_1.d(M_2 - M_3), \\ 3x^q\ M_2.d(M_3 - M_1), \\ 3x^q\ M_3.d(M_1 - M_2) \end{array}\right\} (q=0,\dots,n-1).$$

proof: By a direct computation. ⊠

Consider the surface X_n defined by $F := \Delta_1^2 + \Delta_2^2 + \Delta_3^2 = 0$, which has as normalization $\tilde{X}_n$ an n-star singularity. We determine the semi - universal deformation of $\tilde{X}_n$ by computing essentially the semi - universal admissible deformation of $\Sigma_n \hookrightarrow X_n$. To formulate the result we need some more notation.

Definition (6.14) :

i) Let be given a natural number $n \geq 1$ and $P = \sum_{k \geq 0} p_k . x^k \in \mathbb{C}[x]$. We define the *bracket* $[P]$ as $[P] := \sum_{k \geq n} p_k . x^{k-n}$.

ii) Let $\mathbb{C}[a_1, a_2, a_3, e, b]$ the polynomial ring in the *coefficients* of polynomials a_1 , a_2 , a_3 , $e \in \mathbb{C}[x]$ of degree n-1 and $b \in \mathbb{C}[x]$ of degree n-2. (So it is a polynomial ring in 5n-1 indeterminates and if n=1 there is no b.)

iii) Consider the following conditions on a_i , e and b :

$$e.a_j - [e.a_j].b + [[e.a_j].b].b - [[[e.a_j].b].b].b + \dots = 0 \mod x^n$$

(The dot is multiplication of polynomials and j = 1, 2, 3.)
Note that this a priori infinite series in fact breaks off, so these conditions indeed lead to polynomials in the coefficients of the a_j , e and b. Let J_n be the ideal in $\mathbb{C}[a_1, a_2, a_3, b, e]$ generated by these polynomials.

iv) Let $B(n) := \mathrm{Spec}(\mathbb{C}[a_1, a_2, a_3, b, e]/J_n)$.

Theorem (6.15) :

i) The base space C(n) of a semi-universal deformation of the n-star singularity $\tilde{X}_n$ is isomorphic to $B(n) \times \mathbb{C}$ $(n \geq 2)$ or $B(1)$ $(n=1)$. (The factor $\mathbb{C}$ corresponds to the cross-ratio of the four points on the central $\mathbb{P}^1$ in the resolution graph of the n-star.)

ii) A projection of a semi-universal family for $\tilde{X}_n$ over C(n) is given by the admissible family defined by the equation

$$\tilde{\Delta} . (\tilde{\Delta} + \alpha + \lambda . \beta) = 0$$

where (remember the vector notation and summation conventions.):

* $\tilde{\Delta} = \Delta + \sum_{i=1}^{3} a_i.A_i + b.B + \rho$.

  a_i and b as in (6.14)ii).

  A_i and B as in (6.13)i).

  $\rho = (a_4^2 + a_1.a_2 + a_2.a_3 + a_3.a_1).\ (1,1,1)$

  ($\tilde{\Delta}$ describes the semi-universal deformation of Σ_n.)

* $\alpha = \sum_{i=1}^{\infty} \alpha_i$.

  $\alpha_1 = e.(M_1, M_2, M_3)$.

  $\alpha_2 = e.b.(1,1,1) + \sum_{i=1}^{3} 2.e.a_i.V_i - \sum_{i=1}^{3} [e.a_i].W_i$.

  $\alpha_3 = [e.(a_1.a_2 + a_2.a_3 + a_3.a_1)](1,1,1) - \sum_{i=1}^{3} 2.[e.a_i].b.V_i +$
  $+ \sum_{i=1}^{3} [[e.a_i].b].W_i$.

  $\alpha_{i+1} = -[\alpha_i.b]$ for $i \geq 3$.

  e and b as in (6.14)ii).

  $V_1 = (0,-1,1),\ V_2 = (1,0,-1),\ V_3 = (-1,1,0)$.

  $W_1 = (0,-(M_1+M_2), M_1+M_3)$, $W_2 = (M_1+M_2, 0, -(M_2+M_3))$,

  $W_3 = (-(M_1+M_3), M_2+M_3, 0)$.

* The a_i, e and b satisfy the equations for B(n) (6.14)iv).

* $\lambda \in \mathbb{C}$ is the equisingular parameter (absent if n=1).

  $\beta = (M_1 M_2, 0, 0)$.

We omit the proof, which consists of many pages of straightforward but very tedious computations. Especially we would like to thank J. de Jong for his enthousiastic persistence to complete the computation. A first step in the computation consists of writing down an admissible first order deformation which gives precisely T_X^1 . Then one proceeds along the lines of (3.23). (Note that example (3.24) is the same as the above one with n=1.) It is our experience that the bases chosen in (5.22) are quite convenient, but there might be better ones. In the deformation of Σ_n there appear at most quadratic terms. This one of the reason that after three steps an inductive pattern for the α_i emerges. ⊠

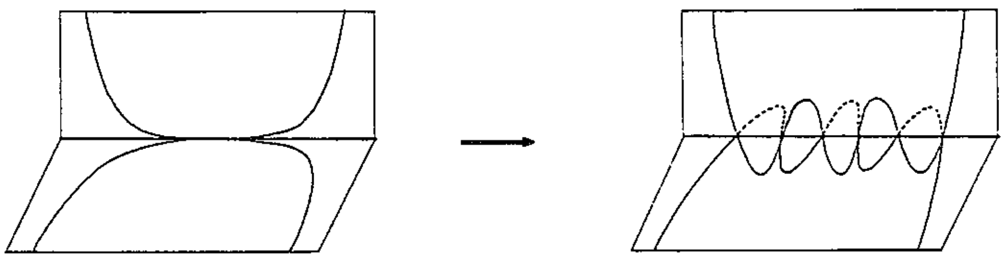
Remark (6.16) :

It is easy to see two of the components of the space B(n). The first one is obtained by putting $e=0$. Then the a_i and b can be arbitrary, so it has dimension $4n-1$. This corresponds to deformations 'which stay in I^2' and gives rise to the Artin component. A second component is obtained by putting the a_i equal to zero and thus has dimension $2n-1$. These deformations include the ones for which Σ_n is not deformed at all. Note further that the equations for B(n) are *linear* in the a_i. We expect that the space B(n) is flat over the b-parameter. For b=0 the equations for B(n) define linear spaces with certain multiplicities. These facts suggest the truth of conjecture (6.1).

We conclude this paragraph by giving some adjacencies for rational quadruple points which are quite easy to see when projected into $\mathbb{C}^3$, but which are, so far as we know, not so easy to see in an other way.

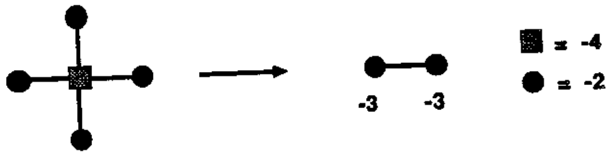
Examples (6.17) :

i) The admissible deformation of $\Sigma_n \hookrightarrow X_n$ described by the equation $(\Delta + t.B + t^2(1,1,1))^2 = 0$, gives for generic t values a surface which has as normalization a surface with n singular points isomorphic the cone over the rational normal curve of degree 4. Indeed, the deformation of the singular locus looks like:



We can perform an additional deformation of this surface in such a way that at p of those n special points we get a triple point whereas at the other n-p points we smooth out the curve. This leads to deformations of the n-star singularity having precisely p triple points $(0 \le p \le n)$. Hence B(n) has at least n+1 components.

ii) J. Wahl describes the following deformation [Wa1]:



When projected to $\mathbb{C}^3$, it can be realized by the admissible deformation given by the equation

$$f_t = (yz)^2 + (y(y-x^2+tx))^2 + (z(z-x^2+tx))^2 + 2.txyz(y+z-x^2+tx) = 0.$$

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